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Albert H. Neubauer Jr.,

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The author would also like to thank the Computer Science Laboratory for the extended use of the IBM 7090 and the United States Air Force, which, through the Institute of Technology program, has made this study possible.

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LIST OF SYMBOLS

A Pin at left end of the connecting rod.

B Pin at right end of the connecting rod.

E Modulus of elasticity.

I Moment of inertia of the cross-sectional area of the connecting rod about an axis orthogonal to x and y.

I_x Longitudinal component of the inertia force of the connecting rod. See equation (6)

I_y Transverse component of the inertia force of the connecting rod. See equation (7)

I_p Inertia force of the piston.

K Positive constant defined in Equation (18).

L Length of the connecting rod.

m_3 Mass per unit length of the connecting rod.

M_3 Mass of the connecting rod.

M_4 Mass of the piston.

N Normal force acting of the piston.

P Force acting of the piston expressed as a function of time.

Q Positive constant defined in Appendix B.

r Length of the crank arm.

R_x Component of the reaction at pin A in the x direction.

R_y Component of the reaction at pin A in the y direction.

t Time.

T Kinetic Energy.

V Potential Energy.

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LIST OF SYMBOLS(cont)

w Displacement from the rigid body position due to vibration in the connecting rod.

$\dot{w}$ Vibrational velocity.

$\ddot{w}$ Vibrational acceleration.

x Coordinate directed along the connecting rod.

X Coordinate directed along the line connecting O and B.

y Coordinate directed perpendicular to the connecting rod.

Y Coordinate orthogonal to X.

$\ddot{\alpha}$ Component of the rigid body acceleration in the x direction.

$\ddot{\beta}$ Component of the rigid body acceleration in the y direction.

φ Angle between the connecting rod and the X coordinate axis.

$\dot{\varphi}$ Angular velocity of the connecting rod.

$\ddot{\varphi}$ Angular acceleration of the connecting rod.

θ Input crank angle expressed as a function of time.

$\dot{\theta}$ Angular velocity of the input crank.

$\ddot{\theta}$ Angular acceleration of the input crank.

ω Constant angular velocity of the input crank. $\theta = \omega t$

ω_n Natural frequency for the n^{th} mode of vibration.

ξ Dummy variable of a general point on the connecting rod in the x direction.

ABSTRACT

Neubauer, Albert H., Jr., Ph.D., Purdue University, June 1963. The Dynamics of an Elastic Linkage. Major Professors: A. S. Hall Jr, and Raymond Cohen.

As the operational speed and accuracy of machinery increases, the method of rigid body linkage analysis fails to adequately predict the dynamic behavior of the members. As a result the exact position of a member at any given time cannot be determined.

~~The purpose of this investigation was to complete~~ an analytical study of the transverse vibrational characteristics of the elastic connecting rod in an in-line slider -crank mechanism is reported. A non-linear fourth order partial differential equation for the transverse vibration of the connecting rod has been derived. This equation was reduced to a linear form involving coefficients which were functions of the position coordinate on the connecting rod and time. This linear equation was solved by a finite difference scheme on an IBM 7090 digital computer.

In addition, a further simplifying assumption was applied to the above linear equation of motion which reduced the coefficients to functions of time only. This simplified linear equation was also solved, using the same finite

difference scheme as above, to determine the validity of this assumption.

The simplified linear fourth order equation of motion was, in turn, reduced to a set of uncoupled second order equations of motion by making use of the technique of normal mode expansions. These uncoupled equations were solved on the IBM 7090 using the Runga-Kutta method of numerical quadratures..

Correlation between the various solutions was achieved. The solution of the first uncoupled equation of motion was found to be a periodic approximation of the finite difference solutions.

The two dimensionless parameters, S the speed parameter and Z the physical parameter, that naturally arise during the derivation of the uncoupled equations of motion can be used by the designer as an indication of the vibrational activity of the mechanism. It was found that Z must always be less than $\pi^2 S$ and that the limiting condition, $Z = \pi^2 S$, is to be avoided.

INTRODUCTION

In the past the bulk of the design work requiring linkage synthesis has been accomplished by assuming that the members or links were rigid bodies; or in other words that no elastic deformation takes place under dynamic loading conditions. Thus for a single degree of freedom system, such as a slider-crank mechanism only a geometrical relationship between the input and the output is required and, therefore, once the input is specified the output is completely determined. If, on the other hand, the connecting rod is assumed to be elastically deformed because of the dynamic loads inherent in/body motion, then the relationship mentioned above becomes inadequate to describe the output as a function of the input and other relationships are required. The output will however execute a perturbative motion about the predicted rigid body position. This effect would, therefore, seem to present a problem in mechanism design if these vibrations were large enough in amplitude to cause the output to vary beyond some arbitrarily chosen acceptable limit. This problem is especially pointed up in high speed, accurate machinery where dynamic loads are high and members have low cross-sectional area to length ratios.

It will be shown that, in general, the situation reduces itself to a complex problem in beam vibration theory. All

but the very basic vibrations texts treat the basic theory of the free transverse and axial vibration of a beam and mathematical solutions are presented for various boundary and initial conditions. Some more advanced texts[1*] extend this theory to consider the effect of axial and transverse loads on the equation of motion and the natural frequencies of the system. But these considerations, although important in the understanding of the problem defined in this thesis, are not directly applicable to the synthesis of a mechanism. Two technical papers have been written applying vibration theory to mechanism synthesis. One, dealing with the longitudinal vibration of the coupler of a crank-rocker mechanism was written by Halit Kosar[2] at the University of Istanbul. This paper derives a non-homogeneous Mathieu equation to represent the differential relationship governing the variation in output crank angle as a function of the input crank angle. Solutions of this equation of motion are compared with the results from an experimental set-up. The other, written by Prof. Dr. Ing. W. Meyer zur Capellen[3], deals with the transverse vibration of the coupler of a crank-rocker mechanism. A method of approach quite similar to that used in this thesis was used by Prof. Meyer zur Capellen to derive a non-homogeneous fourth order partial differential equation of motion with non-constant coefficients. This equation of

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*Numerical superscripts refer to entries in the bibliography.

motion compares with equation (20) of this thesis, and represents the relationship between the magnitude of the transverse vibration of the coupler, the position along the coupler, and the angular input of the crank. A major point of divergence between this investigation and that of Prof. Meyer zur Capellen's occurs in the derivation of the pin reaction relationships. The later of the two authors mentioned above has published many papers on subjects closely related to problems associated with elasticity in linkage design and is probably the foremost authority on this subject.

When approaching the problem of vibration in the connecting rod of a slider-crank mechanism many questions naturally arise which, in the course of an investigation, should be answered. In this case one of the first questions is, under what conditions are vibrations in the connecting rod important? Also, which vibration has more effect on the output, transverse or longitudinal vibrations? If the investigation is to have future use as a design tool, then a relatively simple method of determining the effect of connecting rod vibrations on the output is required, and also the effect that linkage parameters have on these vibrations. To have confidence in the validity of final theoretical mathematical results some correlation between theoretical and experimental data should be achieved. Finally, the results should be extended, if possible, to include more complex data such as input functions that better approximate actual input conditions.

Along with answering some of the above questions, the object of this investigation is to:

a) Determine the equation of motion for the transverse vibration of the connecting rod of a slider-crank mechanism in its most general form.

b) Obtain a solution to the equation of motion by restricting the problem through appropriate simplifying assumptions.

c) Determine the effect of varying linkage parameters on this solution.

ANALYSIS OF THE SYSTEM

Description of the Mechanism

During this investigation only an in-line slider-crank mechanism will be considered. See Figure 1. It will be assumed that the connecting rod of this mechanism is a beam that obeys the fundamental laws of elasticity. For the purpose of simplification, the connecting rod cross-sectional area will be constant and the properties of the rod will not vary along the length.

The geometrical equations governing the motion of this type of slider-crank mechanism are well known, when all of the links are assumed to be rigid bodies. See Appendix A. If the input is taken to be the angular displacement, $\theta(t)$, of the crank and the output is considered to be the position of the piston, X_B, then

$$X_B = r \cos \theta(t) + [L^2 - r^2 \sin^2 \theta(t)]^{\frac{1}{2}} \qquad (1)$$

is the only equation required to determine the input-output relationship. If, on the other hand, the connecting rod is assumed to have elastic properties then the length, L, of this rod will no longer be constant but also an undetermined function of time. The above equation (1), therefore, is no longer adequate to completely describe the system.

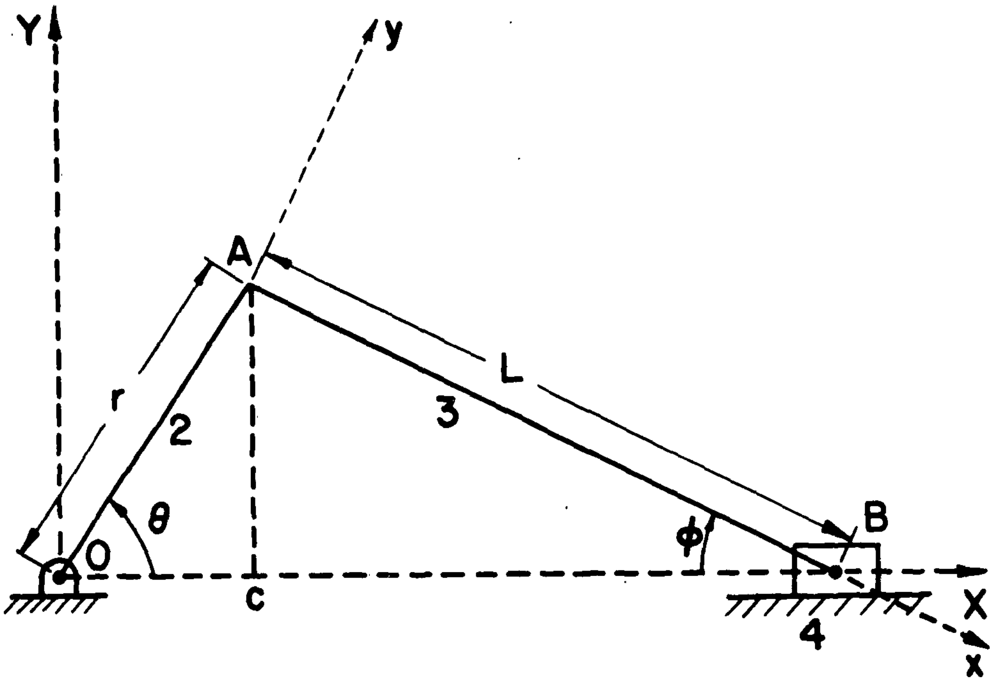

Figure 1. In-Line Slider-Crank Mechanism

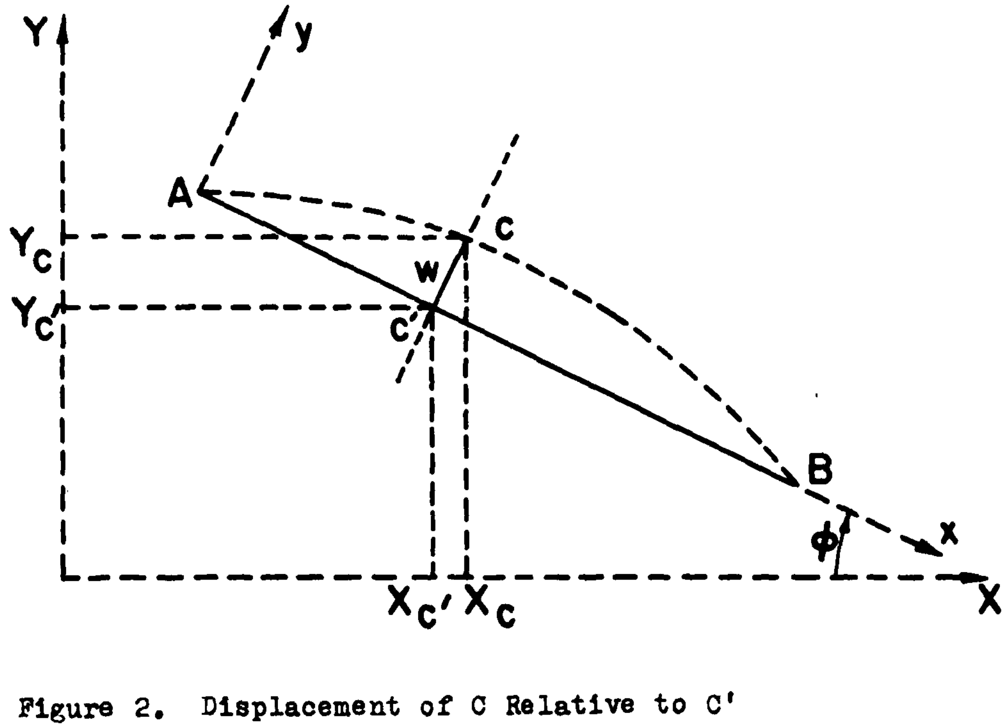

Figure 2. Displacement of C Relative to C'

An acceleration analysis of the rigid body mechanism can easily be made to determine the components of acceleration perpendicular and parallel to the connecting rod for any point on the connecting rod (see Appendix A) as a function of time and displacement along the rod. At the point B, or at $x = L$, the components of equation (2a) perpendicular and parallel to the connecting rod will be equal to the derived expressions, equations (3a) and (4a). Using this method of analysis, equations for the components of acceleration along the connecting rod and perpendicular to the connecting rod can be obtained. These components will be called the rigid body components of acceleration. However, when the assumption of rigid body mechanics is relaxed acceleration components, thus derived, do not represent the accelerations in the system but must be modified to include the accelerations due to vibration.

If a general point C on the connecting rod is assumed to vibrate only in a line perpendicular to the rigid body position C' of this same point on the connecting rod, then from Figure 2, it can be seen that the position of this point in terms of the rigid body position and the magnitude of vibration will be,

$$X_C = X_{C'} + w \sin \varphi$$

$$Y_C = Y_{C'} + w \cos \varphi$$

The first derivative with respect to time yields the velocity

$$\dot{X}_C = \dot{X}_{C'} + \dot{w}\sin\varphi + w\dot{\varphi}\cos\varphi$$
$$\dot{Y}_C = \dot{Y}_{C'} + \dot{w}\cos\varphi - w\dot{\varphi}\sin\varphi \tag{2}$$

The second derivative with respect to time yields the acceleration

$$\ddot{X}_C = \ddot{X}_{C'} + \ddot{w}\sin\varphi + 2\dot{w}\dot{\varphi}\cos\varphi + w\ddot{\varphi}\cos\varphi - w\dot{\varphi}^2\sin\varphi$$
$$\ddot{Y}_C = \ddot{Y}_{C'} + \ddot{w}\cos\varphi - 2\dot{w}\dot{\varphi}\sin\varphi - w\ddot{\varphi}\sin\varphi - w\dot{\varphi}^2\cos\varphi \tag{3}$$

Transforming from the (X,Y) coordinate system to the (x,y) coordinate with the orthogonal transformation equation*

$$\begin{Bmatrix} \ddot{x}_C \\ \ddot{y}_C \end{Bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} \ddot{X}_C \\ \ddot{Y}_C \end{Bmatrix}$$

yields the components of acceleration parallel and perpendicular to the connecting rod

$$\begin{aligned} \ddot{x}_C &= [\ddot{X}_{C'}\cos\varphi - \ddot{Y}_{C'}\sin\varphi] + 2\dot{w}\dot{\varphi} + w\ddot{\varphi} \\ &= \ddot{\alpha}(x,t) + 2\dot{w}(x,t)\,\dot{\varphi}(t) + w(x,t)\,\ddot{\varphi}(t) \end{aligned} \tag{4}$$

$$\begin{aligned} \ddot{y}_C &= [\ddot{X}_{C'}\sin\varphi + \ddot{Y}_{C'}\cos\varphi] + \ddot{w} - w\dot{\varphi}^2 \\ &= \ddot{\beta}(x,t) + \ddot{w}(x,t) - w(x,t)\,\dot{\varphi}^2(t) \end{aligned} \tag{5}$$

In equation (4) $\ddot{\alpha}(x,t)$ represents the rigid body acceleration of the general point C in the x direction; $2\dot{\varphi}\,\dot{w}$

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*Care must be exercised when using this coordinate transformation because the transformation matrix is a function of time.

represents the Coriolis' component of acceleration and $w\ddot{\varphi}$ represents the relative tangential acceleration between points C and C'. In equation (5) $\ddot{\beta}(x,t)$ represents the rigid body acceleration of point C in the y direction; $\ddot{w}(x,t)$ represents the vibrational acceleration, and $w\,\dot{\varphi}^2$ represents the relative normal acceleration between points C and C'. See Appendix A for the derivation of the expressions for $\ddot{\alpha}(x,t)$, equation (3a), and $\ddot{\beta}(x,t)$, equation (4a).

All of the forces acting on the connecting rod and piston are shown in the free body diagram of Figure 3. The pin reactions at the left end, A, and the normal force acting on the piston are unknown. Also unknown is the effect of the vibrational acceleration on the transverse and longitudinal inertia components. In order to derive expressions for these unknowns as well as the eventual derivation of the equation of motion the following assumptions were made:

1) The crank and piston are rigid links and the connecting rod, link 3, is a uniform elastic member.

2) All of the physical properties of the mechanism, the input $\theta(t)$, and the force on the piston, $P(t)$, are known quantities.

3) Friction at the pins A and B and on the slider is assumed zero.

4) The vibration of the connecting rod is small in comparison to the length of the connecting rod.

5) The vibration of the connecting rod is a perturbation

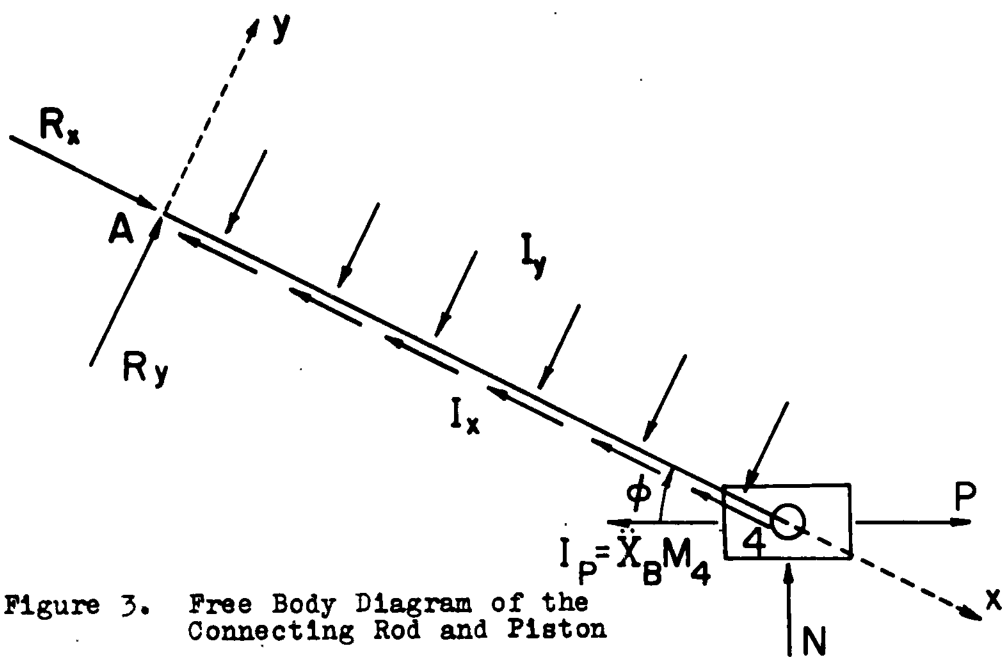

Figure 3. Free Body Diagram of the Connecting Rod and Piston

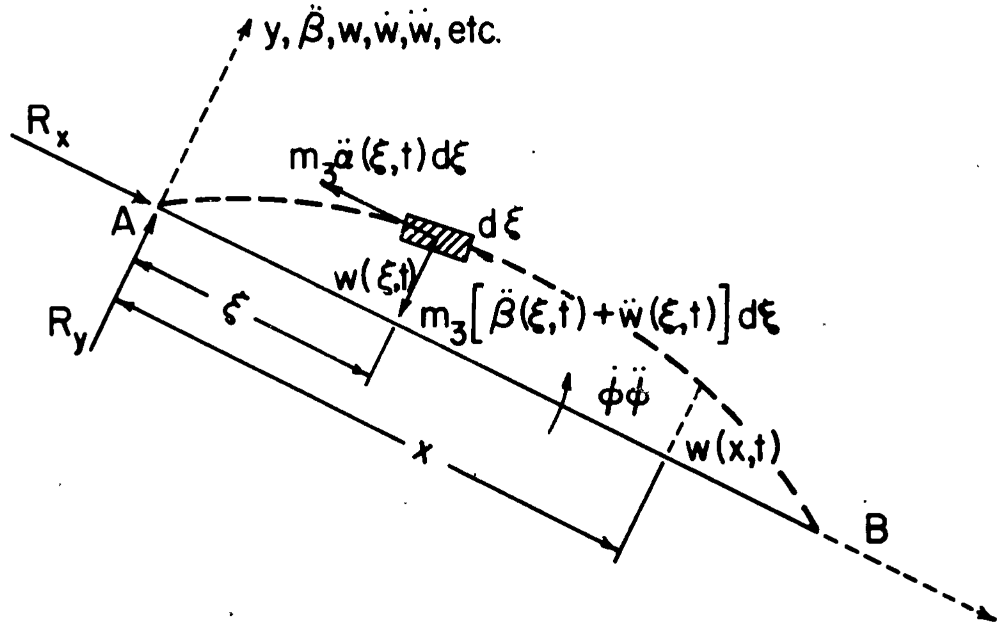

Figure 4. Sign Convention Diagram

about the rigid body motion of link 3.

6) The connecting rod will vibrate only in the transverse direction and the transverse acceleration of the connecting rod will be $\ddot{\beta}(x,t) + \ddot{w}(x,t)$. See equation (5). The normal relative acceleration, $w\,\dot{\varphi}^2$, term is neglected due to assumption 4.

7) The axial acceleration of the connecting rod is $\ddot{\alpha}(x,t)$ only. See equation (4). The Coriolis' component of acceleration, $2\dot{w}\,\dot{\varphi}$, and the tangential relative acceleration, $w\,\ddot{\varphi}$, terms are neglected due to assumption 4. Axial vibration is also neglected.

Using the modified expressions for the longitudinal and transverse acceleration of link 3 the longitudinal and transverse inertia terms I_x and I_y as shown in Figure 3 can now be defined as:

$$I_x = \int_0^L m_3\, \ddot{\alpha}(\xi,t)\, d\,\xi \tag{6}$$

$$I_y = \int_0^L m_3 [\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\, d\,\xi. \tag{7}$$

Figure 4 shows the sign conventions that will be used during the derivation of the equation of motion. In general, the positive x and y axes define positive directions, except that positive inertia forces are directed opposite to positive acceleration.

Derivation of the Equation of Motion

The first step in the derivation of the equation of motion is to determine the pin reactions R_x and R_y. Making use of the D'Alembert principle of dynamic equilibrium and remembering that a positive inertia force is in the direction of negative acceleration, the following force and moment summations can be made

$$\Sigma F_x = R_x - I_x - N \sin \varphi + (P-I_p) \cos \varphi = 0, \tag{8}$$

$$\Sigma F_y = R_y - I_y + N \cos \varphi + (P-I_p) \sin \varphi = 0, \tag{9}$$

$$\Sigma M_B = -L\, R_y + \int_0^L m_s\, w(\xi,t)\, \ddot{\alpha}(\xi,t)\, d\,\xi + \int_0^L m_s (L-\xi)[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\, d\,\xi = 0. \tag{10}$$

From (10) the reaction R_y can be determined as

$$R_y = \int_0^L \frac{m_s}{L}\, w(\xi,t)\, \ddot{\alpha}(\xi,t)\, d\,\xi + \int_0^L m_s (1-\frac{\xi}{L})[\ddot{\beta}(\xi,t)+\ddot{w}(\xi,t)]\, d\,\xi. \tag{11}$$

Substituting (11) into (9) and solving for N gives

$$N = \frac{1}{\cos \varphi} [-\int_0^L \frac{m_s}{L}\, w(\xi,t)\, \ddot{\alpha}(\xi,t)\, d\,\xi + \int_0^L m_s \frac{\xi}{L} [\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\, d\,\xi - (P-I_p) \sin \varphi]. \tag{12}$$

Substituting (12) into (8) then gives the remaining pin reaction

$$R_x = \int_0^L m_s [1 - \frac{w(\xi,t) \tan\varphi}{L}] \ddot{\alpha}(\xi,t)\, d\xi +$$

$$\int_0^L m_s \frac{\xi}{L} \tan\varphi [\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\, d\xi - (P-I_p) \sec\varphi. \quad (13)$$

With the required pin reactions R_x and R_y now given as known functions, the Euler-Bernoulli equation

$$- E\, I\, w''(x,t) = \Sigma\, M(x,t)$$

can be employed to derive the equation of motion. In the above equation, the negative sign appears because counter-clockwise moments, which are assumed positive, cause negative curvature. The prime notation denotes differentiation with respect to x.

Summing moments about some general point x, see Figure 4, and substituting this summation into the above equation gives

$$- E\, I\, w''(x,t) = R_x\, w(x,t) - R_y\, x +$$

$$\int_0^x m_s (x-\xi)[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\, d\xi -$$

$$\int_0^x m_s\, [w(x,t) - w(\xi,t)]\, \ddot{\alpha}(\xi,t)\, d\xi. \quad (14)$$

Substituting for R_x and R_y the expression in equations (11) and (13) gives the complete form of the Euler-Bernoulli equation

$$-E\,I\,w''(x,t) = w(x,t)\left[\int_0^L [1 - \frac{w(\xi,t)\tan\varphi}{L}]\,\ddot{\alpha}(\xi,t)\,d\,\xi + \right.$$

$$\left.\int_0^L m_3\,\frac{\xi}{L}\tan\varphi[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi - (P\text{-}I_p)\sec\varphi\right] -$$

$$x\left[\int_0^L \frac{m_3}{L}\,w(\xi,t)\,\ddot{\alpha}(\xi,t)\,d\,\xi + \int_0^L m_3(1-\frac{\xi}{L})[\ddot{\beta}(\xi,t)+\ddot{w}(\xi,t)]d\xi\right]$$

$$+ \int_0^x m_3(x-\xi)[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi -$$

$$\int_0^x m_3[w(x,t) - w(\xi,t)]\,\ddot{\alpha}(\xi,t)\,d\,\xi.$$

Rearranging terms for convenience yields the equation

$$-E\,I\,w''(x,t) = -\,w(x,t)[P-I_p]\sec\varphi -$$

$$\int_0^x m_3[w(x,t) - w(\xi,t)]\,\ddot{\alpha}(\xi,t)\,d\,\xi +$$

$$\int_0^L m_3\,[w(x,t) - \frac{w(x,t)\,w(\xi,t)\tan\varphi}{L} - \frac{x\,w(\xi,t)}{L}]\,\ddot{\alpha}(\xi,t)d\xi$$

$$+ \int_0^x m_3(x-\xi)[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi +$$

$$\int_0^L m_3[\frac{\xi w(x,t)\tan\varphi}{L} - x(1-\frac{\xi}{L})][\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi. \quad (15)$$

In an attempt to eliminate the integrals in the above equation (15) the standard technique of using Leibnitz' rule for the differentiation of an integral will be used. Therefore, differentiating equation (15) with respect to x gives

$$-E\,I\,w'''(x,t) = -\,w'(x,t)[P-I_P]\sec\varphi -$$

$$w'(x,t)\int_0^x m_3\,\ddot{\alpha}(\xi,t)\,d\,\xi +$$

$$\int_0^L m_3[w'(x,t)\{1-\frac{w(\xi,t)\tan\varphi}{L}\} - \frac{w(\xi,t)}{L}]\,\ddot{\alpha}(\xi,t)\,d\,\xi +$$

$$\int_0^x m_3[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi +$$

$$\int_0^L m_3[\frac{\xi w'(x,t)\tan\varphi}{L} - (1-\frac{\xi}{L})][\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi.$$

Differentiating again with respect to x gives

$$-E\,I\,w^{IV}(x,t) = -w''(x,t)[P-I_P]\sec\varphi - w''(x,t)\int_0^x m_3\ddot{\alpha}(\xi,t)\,d\,\xi$$

$$+\,w''(x,t)\int_0^L m_3[1-\frac{w(\xi,t)\tan\varphi}{L}]\,\ddot{\alpha}(\xi,t)\,d\,\xi -$$

$$w'(x,t)[m_3\ddot{\alpha}(x,t)] +$$

$$w''(x,t)\int_0^L m_3\,\frac{\xi\tan\varphi}{L}\,[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi +$$

$$m_3[\ddot{\beta}(x,t) + \ddot{w}(x,t)]. \qquad (16)$$

Without looking at the functions which represent the coefficients the above equation (16) can be written in the following form

$$w^{IV}(x,t) + f_1(x,t)\,w''(x,t) + f_2(x,t)\,w'(x,t) + A\,\ddot{w}(x,t) =$$

$$w''(x,t)\int_0^L[f_3(\xi,t)\ddot{w}(\xi,t) + f_4(\xi,t)w(\xi,t)]\,d\,\xi + f_5(x,t).$$

The last equation can be classified as a non-linear, non-homogeneous, fourth order partial integral-differential equation with non-constant coefficients. Equations of this type do not lend themselves to complete mathematical solutions or even to solutions using numerical techniques.

Within the frame work of the above equation, however, the fourth order partial differential equation governing the free transverse vibration of a beam

$$w^{IV}(x,t) + A\ w(x,t) = 0$$

can be recognized. The additional terms arise from the forced motion of the beam (connecting rod) through its constrained positions.

Assumptions for Linearization

Since the derived equation of motion (16) is non-linear in character and cannot be linearized by ordinary mathematical techniques, a study of the basic components of the Euler-Bernoulli formulation was undertaken to determine where the non-linear terms arise. By following the various components of the Euler-Bernoulli formulation (14) through the process of differentiation, it can be seen that all of the non-linear terms arise from the derived expression for R_x, equation (13). R_y, not being a function of w or x, drops out during the differentiation with respect to x. A study of R_x was therefore undertaken to determine if this term could be simplified.

If the assumption is make that the end reactions do not depend upon the vibration then all of the non-linear terms

are neglected and the pin reactions become those derived if all links are assumed to be rigid. By eliminating the terms in equation (13) that depend on w or its time derivitives a relatively complex function of time is obtained

$$R_x = \int_0^L m_s\ \ddot{\alpha}(\xi,t)\ d\ \xi + +$$

$$\int_0^L m_3\ \frac{\xi}{L} \tan\varphi\ [\ddot{\beta}(\xi,t)]\ d\ \xi - (P-I_P)\sec\varphi. \qquad (13')$$

This function (13') can be substituted into equation (14) and after differentiating twice with respect to x a simplified form of equation (16)

$$-E\ I\ w^{IV}(x,t) = -\ w''(x,t)[P-I_P]\sec\varphi\ -$$

$$w''(x,t)\int_0^x m_3\ \ddot{\alpha}(\xi,t)\ d\ \xi\ +$$

$$w''(x,t)\int_0^L m_3\ \ddot{\alpha}(\xi,t)\ d\ \xi - w'(x,t)\ m_s\ \ddot{\alpha}(x,t)\ +$$

$$w''(x,t)\int_0^L \frac{m_3\,\xi\,\tan\varphi}{L}\ \ddot{\beta}(\xi,t)\ d\ \xi\ +$$

$$m_s[\ddot{\beta}(x,t) + \ddot{w}(x,t)] \qquad (16')$$

is obtained. This equation (16') is a non-homogeneous, linear, fourth order partial differential equation with non -constant coefficients that can be solved on a high speed digital computer by using numerical techniques.

However, in the hope of further reducing the complexity of equation (16') the rigid body force analysis of Appendix

B was undertaken. The result of this analysis is the assumption that

$$R_x = - K \cos \theta(t). \tag{17}$$

Where K is a positive constant that can be approximated by

$$K = (M_3 + M_4)\, r\, \omega^2 . \tag{18}$$

It must be noted that, in order to make this simplifying assumption, the scope of the investigation was restricted to mechanisms with small r/L and M_3/M_4 ratios and to input functions that are linear in time, $\theta=\omega t$. If these restrictions are unsuitable then equation (16') must be solved directly.

A further simplifying assumption arises from the inspection of the last term in equation (14). In light of assumption 4 on page 9, it can be noted that the term

$$w(x,t) - w(\xi,t)$$

is proportional to the slope, w'. Therefore, in light of the required assumption made during the derivation of the Euler-Bernoulli equation that the slope must be small, the term written above can be neglected. Assuming that this term equals zero should not introduce into the problem an error of magnitude greater than that introduced previously by assuming that the slope is zero. This assumption, therefore, keeps the right and left hand sides of equation (14) within the same order of magnitude.

Derivation of Simplified Equations of Motion

If the function representing R_x in equation (17) is substituted into the Euler-Bernoulli equation(14) then the equation

$$-E\,I\,w''(x,t) = -\,w(x,t)\,K\cos\theta + x\,R_y + \int_0^x m_s(x-\xi)[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi - \int_0^x m_s[w(x,t) - w(\xi,t)]\,\ddot{\alpha}(\xi,t)\,d\,\xi \qquad (19)$$

results. Employing the same technique, equation (19) is differentiated twice with respect to x

$$-E\,I\,w'''(x,t) = -\,w'(x,t)\,K\cos\theta + R_y + \int_0^x m_s[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi - w'(x,t)\int_0^x m_s\,\ddot{\alpha}(\xi,t)\,d\,\xi\,,$$

$$-E\,I\,w^{IV}(x,t) = -\,w''(x,t)\,K\cos\theta + m_s[\ddot{\beta}(x,t) + \ddot{w}(x,t)] - w''(x,t)\int_0^x m_s\,\ddot{\alpha}(\xi,t)\,d\,\xi - w'(x,t)[m_s\,\ddot{\alpha}(x,t)]. \qquad (20)$$

Equation (20) is the linear form of the equation of motion.

If the assumption, that the last term in equation (19) can be neglected, is now included, equation(19) becomes

$$-E\,I\,w''(x,t) = -\,w(x,t)\,K\cos\theta + x\,R_y + \int_0^x m_s(x-\xi)[\ddot{\beta}(\xi,t) + \ddot{w}(\xi,t)]\,d\,\xi. \qquad (21)$$

Again employing the same technique, equation (21) is differentiated twice with respect to the displacement x and the equation

$$-E\ I\ w^{IV}(x,t) = -\ w''(x,t)\ K \cos \theta + m_s[\ddot{\beta}(x,t)+\ddot{w}(x,t)] \quad (22)$$

results. This equation(22) is the most simplified form of the equation of motion. With the exception of the middle term, equation (22) is the equation of motion for the transverse vibration of a beam with an external forcing function. The middle term in equation (22) represents the effect of a time varying axial end load on the transverse vibration of a beam.

To complete the statement of the mathematical problem four boundary and two initial conditions must be specified. The boundary conditions are the same as those for a beam with pinned ends. In terms of the symbols used these are

$$w(0,t) = 0, \qquad w''(0,t) = 0,$$
$$w(L,t) = 0, \qquad w''(L,t) = 0.$$

These conditions, stated in words, are: The deflection of the beam ends is zero and the moment (second derivative) at these ends is also zero. If the assumption is made that the connecting rod is started from $\theta = 0°$ from rest, the initial conditions are

$$w(x,0) = 0, \qquad \dot{w}(x,0) = 0.$$

These conditions, stated in words, are: The connecting rod is initially a straight beam and the motion is started from a rest position, no initial velocity.

Reduction of the Equation of Motion to a Set of Uncoupled Equations

A standard mathematical technique for obtaining the solution to a partial differential equation is the method of separation of variables. To use this technique the assumption is made that the solution is of the form

$$w(x,t) = \sum_{n=1}^{\infty} q_n(t)\, \varphi_n(x) \tag{23}$$

where $q_n(t)$ is only a function of time and $\varphi_n(x)$ is only a function of displacement. The assumed solution, equation (23), is then substituted into the partial differential equation and the equations governing the functions of displacement, $\varphi_n(x)$, and the functions of time $q_n(t)$, are determined. To mathematically determine these functions it is necessary to prove that, for a vibrating system of this type, the modes of vibration, $\varphi_n(x)$, are orthogonal,[1] that is

$$\int_0^L \varphi_n(x)\, \varphi_m(x) = 0 \qquad m \neq n \tag{24}$$

It has been shown in the literature that, for the free transverse vibration of a beam with pinned ends, the modes of vibration are[1]

$$\varphi_n(x) = \sin \frac{n\pi x}{L} \, . \tag{25}$$

The trigonometric functions are known to be a complete set of orthogonal functions; therefore, the conditions stated in (24) are fulfilled. It can be noted also that the $\varphi_n(x)$ functions satisfy the boundary conditions as stated previously.

For the solution of equation (22) the assumption will, therefore, be made that the solution is of the form

$$w(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L} \tag{26}$$

and the time functions, $q_n(t)$, will be determined.

Substituting equation (26) and its required derivatives with respect to both time and displacement into the equation of motion (22) gives

$$E\,I \sum_{n=1}^{\infty} q_n(t) \left(\frac{n\pi}{L}\right)^4 \sin \frac{n\pi x}{L} + m_3\, \ddot{\beta}(x,t) +$$

$$m_s \sum_{n=1}^{\infty} \ddot{q}_n(t) \sin \frac{n\pi x}{L} +$$

$$K \cos\theta \sum_{n=1}^{\infty} q_n(t) \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} = 0. \tag{27}$$

Making use of the orthogonality condition by multiplying the above equation by $\sin m\pi x/L$ and integrating over the length of the connecting rod, yields the relationship, a second order ordinary differential equation, that the $q_n(t)$ must satisfy. Performing this operation yields

$$EI \int_0^L \sum_{n=1}^{\infty} q_n(t) \left[\frac{n\pi}{L}\right]^4 \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx +$$

$$\int_0^L m_3 \, \ddot{\beta}(x,t) \sin \frac{m\pi x}{L} \, dx + m_3 \int_0^L \sum_{n=1}^{\infty} \ddot{q}_n(t) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx +$$

$$K \cos \theta \int_0^L \sum_{n=1}^{\infty} q_n(t) \left[\frac{n\pi}{L}\right]^2 \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx . \quad (28)$$

Interchanging the sum and integral signs and noting that, from orthogonality

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = 0 \qquad m \neq n$$

and

$$\int_0^L \sin^2 \frac{n\pi x}{L} \, dx = \frac{L}{2}$$

yields (all terms except when n=m are dropped from (28)

$$\frac{m_3 L}{2} \ddot{q}_n(t) + \left[\frac{EI(n\pi)^4}{2L^3} + \frac{(n\pi)^2 K \cos \theta}{2L}\right] q_n(t) =$$

$$-m_3 \int_0^L \ddot{\beta}(x,t) \sin \frac{n\pi x}{L} \, dx$$

or for convenience

$$\ddot{q}_n(t) + \left[\frac{EI(n\pi)^4}{L^4 m_3} + \frac{(n\pi)^2 K \cos \theta}{L^2 m_3}\right] q_n(t) =$$

$$- \frac{2}{L} \int_0^L \ddot{\beta}(x,t) \sin \frac{n\pi x}{L} \, dx \qquad (29)$$

Since $\ddot{\beta}(x,t)$ is a known function, equation (4a'), the integral on the right hand side of equation (29) can be evaluated as a function of time only.

Equations (29) are known as the uncoupled equations of motion and, in this case, are in the form of non-homogeneous Mathieu equations. The natural frequencies for the various modes of vibration are represented by the term

$$\omega_n^2 = \left[\frac{(n\pi)^4\ EI}{L^4 m_3} + \frac{(n\pi)^2\ K \cos\theta}{L^2 m_3}\right].$$

It can be noted that the natural frequencies of this system vary with time. This is to be expected since the axial loads are time varying functions.

To further verify the validity of equations (29) as the uncoupled equations of motion, the Lagrangian approach is employed to derive these equations.

Making use of the same assumptions that were required to arrive at the equation of motion (22) the kinetic and potential energies can be formulated as follows:

$$T = \frac{m_3}{2} \int_0^L [\dot{x}_c^2 + \dot{y}_c^2]\, dx$$

where $\dot{x}_c$ and $\dot{y}_c$ are defined by equation (2) since during an orthogonal transformation the length of a vector remains constant. Therefore,

$$\dot{x}_c^2 + \dot{y}_c^2 = \dot{X}_c^2 + \dot{Y}_c^2 .$$

The transformed set of equations for the velocity of the general point C are not used because the transformation matrix is a function of time and later, when substituting this kinetic energy term into Lagranges' Equation, differentiation with respect to time is required and a term will be lost. The kinetic energy, therefore, is

$$T = \frac{m_s}{2} \int_0^L [(\dot{X}_{c'} + \dot{w} \sin\varphi + w\dot{\varphi}\cos\varphi)^2 +$$

$$(\dot{Y}_{c'} + \dot{w}\cos\varphi - w\dot{\varphi}\sin\varphi)^2]\, dx$$

$$= \frac{m_s}{2} \int_0^L [\dot{X}_{c'}{}^2 + \dot{Y}_{c'}{}^2 + 2\dot{w}\,(\dot{X}_{c'}\sin\varphi + \dot{Y}_{c'}\cos\varphi) +$$

$$2w\,\dot{\varphi}(\dot{X}_{c'}\cos\varphi - \dot{Y}_{c'}\sin\varphi) + \dot{w}^2 + w^2\,\dot{\varphi}^2]\, dx \qquad (30)$$

By inspecting two of the terms in the above equation it can be seen that the coefficient of $2\dot{w}$ is the component of the rigid body velocity of point C' in the y direction and the coefficient of $2w\,\dot{\varphi}$ is the component of the rigid body velocity of point C' in the x direction.

If the form of the solution is again assumed to be the infinite sum of the product of a function of time and a function of displacement then the expression (26) can be substituted into the kinetic energy equation (30). In this situation the $q_n(t)$ are thought of as an infinite set of generalized coordinates that completely describe the motion of the system.

Substituting this assumed form of the solution and its appropriate derivatives into equation (30) yields the final form of the kinetic energy term.

$$T = \frac{m_3}{2}\int_0^L \{\dot{X}_{c'}{}^2 + \dot{Y}_{c'}{}^2 +$$

$$2\sum_{n=1}^{\infty} \dot{q}_n(t) \sin\frac{n\pi x}{L}(\dot{X}_{c'} \sin\varphi + \dot{Y}_{c'} \cos\varphi) +$$

$$2\sum_{n=1}^{\infty} q_n(t) \sin\frac{n\pi x}{L}[\dot{\varphi}(\dot{X}_{c'} \cos\varphi - \dot{Y}_{c'} \sin\varphi)] +$$

$$\sum_{n=1}^{\infty} \dot{q}_n{}^2(t) \sin^2\frac{n\pi x}{L} + \dot{\varphi}^2 \sum_{n=1}^{\infty} q_n{}^2(t) \sin^2\frac{n\pi x}{L}\}\, dx.$$

The potential energy can be written as the sum of the potential energy due to bending and the potential energy due to the axial load.[1]

$$V = \frac{EI}{2}\int_0^L [w''(x,t)]^2\, dx + \frac{K\cos\theta}{2}\int_0^L [w'(x,t)]^2\, dx.$$

Substituting the first and second derivatives with respect to x of the assumed form of the solution into this equation for the potential energy yields

$$V = \frac{EI}{2}\int_0^L \sum_{n=1}^{\infty} q_n{}^2(t)\left[\frac{n\pi}{L}\right]^4 \sin^2\frac{n\pi x}{L}\, dx +$$

$$\frac{K\cos\theta}{2}\int_0^L \sum_{n=1}^{\infty} q_n{}^2(t)\left[\frac{n\pi}{L}\right]^2 \cos^2\frac{n\pi x}{L}\, dx.$$

Interchanging the summation and integral signs and evaluating the integrals reduces this equation to

$$V = \frac{EI\pi^4}{4L^3} \sum_{n=1}^{\infty} n^4 q_n^2(t) + \frac{\pi^2 K \cos\theta}{4L} \sum_{n=1}^{\infty} n^2 q_n^2(t) .$$

Lagranges' Equations for this situation take the general form

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_n(t)}\right] - \frac{\partial T}{\partial q_n(t)} + \frac{\partial V}{\partial q_n(t)} = 0$$

The first term becomes

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_n(t)}\right] = \frac{d}{dt}\left[m_3 \int_0^L (\dot{X}_{c'} \sin\varphi + \dot{Y}_{c'} \cos\varphi) \sin\frac{n\pi x}{L}\, dx + \right.$$

$$\left. \dot{q}_n(t) \int_0^L m_3 \sin^2\frac{n\pi x}{L}\, dx\right]$$

$$= m_3 \int_0^L \{(\ddot{X}_{c'} \sin\varphi + \dot{X}_{c'}\dot{\varphi} \cos\varphi + \ddot{Y}_{c'} \cos\varphi -$$

$$\dot{Y}_{c'}\dot{\varphi} \sin\varphi) \sin\frac{n\pi x}{L} + \ddot{q}_n(t) \sin^2\frac{n\pi x}{L}\}\, dx .$$

By evaluating the integral wherever possible and by replacing those terms that represent the transverse component of the rigid body acceleration with $\ddot{\beta}(x,t)$ the above becomes

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_n(t)}\right] = m_3 \int_0^L \ddot{\beta}(x,t) \sin\frac{n\pi x}{L}\, dx + \frac{m_3 L}{2} \ddot{q}_n(t) +$$

$$m_3 \int_0^L \dot{\varphi}(\dot{X}_{c'} \cos\varphi - \dot{Y}_{c'} \sin\varphi) \sin\frac{n\pi x}{L}\, dx .$$

The second term becomes

$$\frac{\partial T}{\partial q_n(t)} = m_3 \int_0^L \dot{\varphi}(\dot{X}_{c'} \cos \varphi - \dot{Y}_{c'} \sin \varphi) \sin \frac{n\pi x}{L} \, dx +$$

$$\frac{m_3 L\dot{\varphi}^2}{2} q_n(t) .$$

The third term becomes

$$\frac{\partial V}{\partial q_n(t)} = \frac{n^4\pi^4 EI}{2L^3} q_n(t) + \frac{n^2\pi^2 \ K \cos \theta}{2L} q_n(t) .$$

Combining these terms yields the uncoupled equations of motion

$$\frac{m_3 L}{2} \ddot{q}_n(t) + m_3 \int_0^L \ddot{\beta}(x,t) \sin \frac{n\pi x}{L} \, dx +$$

$$m_3 \int_0^L \dot{\varphi}(\dot{X}_{c'} \cos \varphi - \dot{Y}_{c'} \sin \varphi) \sin \frac{n\pi x}{L} \, dx -$$

$$m_3 \int_0^L \dot{\varphi}(\dot{X}_{c'} \cos \varphi - \dot{Y}_{c'} \sin \varphi) \sin \frac{n\pi x}{L} \, dx -$$

$$\frac{m_3 L \ \dot{\varphi}^2}{2} q_n(T) + \frac{n^4\pi^4 \ EI}{2L^3} q_n(t) + \frac{n^2\pi^2 \ K \cos \theta}{2L} q_n(t) = 0 .$$

Rearranging terms for comparison purposes and canceling the third and fourth terms yields

$$\ddot{q}_n(t) + \left[\frac{n^4\pi^4 EI}{L^4 m_3} + \frac{n^2\pi^2 \ K \cos \theta}{L^2 m_3} - \dot{\varphi}^2\right] q_n(t) = -$$

$$\frac{2}{L} \int_0^L \ddot{\beta}(x,t) \sin \frac{n\pi x}{L} \, dx . \qquad (33)$$

Equation (33) agrees with equation (29) term for term except for the third term in the coefficient of $q_n(t)$. This term does not appear in equation (29) because of the assumption that the normal relative acceleration term in the transverse component of acceleration can be neglected. See assumption 6 on page 11.

SOLUTIONS OF THE EQUATIONS OF MOTION

Uncoupled Equations

As derived in the preceding section the uncoupled equations of motion are

$$\ddot{q}_n(t) + \left[\frac{n^4\pi^4 EI}{L^3 M_3} + \frac{n^2\pi^2}{L}\frac{K\cos\theta}{M_3}\right] q_n(t) = -$$

$$\frac{2}{L}\int_0^L \ddot{\beta}(x,t)\sin\frac{n\pi x}{L}\,dx. \tag{29}$$

Substituting the derived expression (4a') for $\ddot{\beta}(x,t)$ into equation (29) and evaluating the integral on the right hand side reduces this equation to

$$\ddot{q}_n(t) + \left[\frac{n^4\pi^4 EI}{L^3 M_3} + \frac{n^2\pi^2}{L}\frac{K\cos\theta}{M_3}\right] q_n(t) = -$$

$$\frac{2g_3(t)}{n\pi}\left[1-\cos n\pi\right] - \frac{2L\;g_4(t)}{n\pi}\left[-\cos n\pi\right]. \tag{34}$$

Where $g_3(t)$ and $g_4(t)$ are defined in equations (4a') and (4a") of Appendix A; K is defined in equation (18), and $n = 1,2,3,\cdots$.

When n is an odd integer, $n = 1,3,5,\cdots$, equation (34) reduces to

$$\ddot{q}_n(t) + \left[\frac{n^4\pi^4 EI}{L^3 M_3} + \frac{n^2\pi^2}{L}\frac{K\cos\theta}{M_3}\right] q_n(t) = -$$

$$\frac{4}{n\pi}\left[g_3(t) + \frac{L}{2}\, g_4(t)\right] \tag{35}$$

where the right hand side in proportional to the transverse component of acceleration of the center of the connecting rod.

When n is an even integer, $n = 2,4,6,\cdots$, equation (34) reduces to

$$\ddot{q}_n(t) + \left[\frac{n^4\pi^4 EI}{L^3 M_3} + \frac{n^2\pi^2}{L}\frac{K\cos\theta}{M_3}\right] q_n(t) = \frac{2L\, g_4(t)}{n\pi} \tag{36}$$

where the right hand side is proportional to the relative tangential acceleration. Substituting the expressions for $g_3(t)$, $g_4(t)$ and K into equations (35) and (36) yield the complete form of the uncoupled equations of motion

$$\ddot{q}_n(t) + \left[\frac{n^4\pi^4 EI}{L^3 M_3} + n^2\pi^2\left(1+\frac{M_4}{M_3}\right)\frac{r}{L}\,\omega^2\cos\theta\right] q_n(t) = \begin{cases} Q_o & n=1,3,5\cdots \\ Q_e & n=2,4,6\cdots \end{cases}$$

where $Q_o = \frac{4}{n\pi}\Big[r\,\omega^2(\cos\theta\sin\varphi + \sin\theta\cos\varphi) -$

$$\frac{L}{2}\left(\frac{r\omega^2}{L}\,\frac{\sin\theta}{\cos\varphi} - \frac{r^2\omega^2}{L^2}\,\frac{\cos^2\theta\,\sin\varphi}{\cos^3\varphi}\right)\Big] \tag{37}$$

and $$Q_e = \frac{2L}{n\pi}\left[\frac{r\omega^2}{L}\,\frac{\sin\theta}{\cos\varphi} - \frac{r^2\omega^2}{L^2}\,\frac{\cos^2\theta\,\sin\varphi}{\cos^3\varphi}\right]. \tag{38}$$

Since θ, the angular input, is a more desirable variable than time and since there exists a simple relationship between

θ and t, $\theta = \omega t$, a change of variable from t to θ in equations (37) and (38) yields after rearranging terms on the right hand side and remembering that

$$\frac{d^2 q_n}{dt^2} = \omega^2 \frac{d^2 q_n}{d\theta^2}$$

$$\ddot{q}_n(\theta) + \left[\frac{n^4 \pi^4 EI}{L^3 M_3 \omega^2} + n^2 \pi^2 \left(1 + \frac{M_4}{M_3}\right) \frac{r}{L} \cos\theta\right] q_n(\theta) = \begin{cases} Q_o' & n=1,3,5,\cdots \\ Q_e' & n=2,4,6\cdots \end{cases}$$

where $Q_o' = \frac{2r}{n\pi}\Big[2(\cos\theta \sin\varphi + \sin\theta\cos\varphi) -$

$$\frac{\sin\theta}{\cos\varphi} + \frac{r}{L}\,\frac{\cos^2\theta \sin\varphi}{\cos^3\varphi}\Big] \qquad (37')$$

and $Q_e' = \frac{2r}{n\pi}\left[\frac{\sin\theta}{\cos\varphi} - \frac{r}{L}\,\frac{\cos^2\theta\sin\varphi}{\cos^3\varphi}\right]$. $\qquad$ (38')

The dot notation now denotes differentiation with respect to θ.

Inspection of the terms in equations (37') and (38') shows that two dimensionless ratios naturally arise that include all of the parameters of the system. These dimensionless equations are

$$S = \frac{EI}{L^3 M_3 \omega^2} \qquad (39)$$

$$Z = \left(1 + \frac{M_4}{M_3}\right)\frac{r}{L} \qquad (40)$$

S can be thought of as the dimensionless parameter primarily describing the input angular velocity and Z can be thought of

as the parameter that describes the mechanism.

The right hand sides of equations (37') and (38') have not been non-dimensionalized but depend upon the parameter r, the length of the driving crank, and the dimensionless ratio r/L. These equations, however, could be non-dimensionalized by redefining the vibrational amplitude as

$$\bar{q}_n(\theta) = q_n(\theta)/r.$$

The problem would then reduce itself to a study of the effects of varying the three dimensionless parameters r/L, Z and S. Since Z depends upon the r/L ratio, the effects of varying r/L, M_3/M_4 and S on the vibrational activity of the mechanism would be sufficient.

To solve the uncoupled equations of motion the Runga -Kutta method of numerical quadratures[4] was programed into the IBM 7090 high speed digital computer at the Statistical Laboratory at Purdue University.

In order to solve a second order differential equation by this method it is necessary to reduce the differential equation to a system of first order differential equations. This is accomplished by setting

$$\dot{q}_{n_1}(\theta) = q_{n_2}(\theta)$$

$$\dot{q}_{n_2}(\theta) = -\left[n^4\pi^4 S + n^2\pi^2 Z \cos\theta\right] q_{n_1}(\theta) + f_n(\theta) \qquad (41)$$

where

$$f_n(\theta) = \frac{2r}{n\pi}\left[2(\cos\theta\sin\varphi + \sin\theta\cos\varphi) - \frac{\sin\theta}{\cos\varphi} + \frac{r}{L}\frac{\cos^2\theta\sin\varphi}{\cos^3\varphi}\right] n = 1,3,5,\cdots$$

and

$$f_n(\theta) = \frac{2r}{n\pi}\left[\frac{\sin\theta}{\cos\varphi} - \frac{r}{L}\,\frac{\cos^2\theta\sin\varphi}{\cos^3\varphi}\right] \quad n = 2,4,6\cdots$$

Equations (41) in conjunction with initial conditions are solved simultaneously to obtain the desired result, $q_n(t)$. Since it has been assumed that the mechanism will be started from rest, and since $q_n(t)$ is the function that describes the behavior of time, the initial conditions for equations (37') and (38') are

$$q_n(0) = \dot{q}_n(0) = 0$$

or for equations (41) are

$$q_{n_1}(0) = q_{n_2}(0) = 0.$$

Effect of Damping

The above initial conditions may not be desirable in some future investigation of this problem. If, however, damping can be introduced into the equations of motion only the steady state solution will remain after the transients caused by free vibration have been damped out. The uncoupled equations of motion are the only form of the equations of motion thus far derived that yield a relatively simple opportunity to study the effects of damping. These equations of motion are in the well known uncoupled form

$$\ddot{q}_n(t) + \omega_n^2\, q_n(t) = f(t)$$

which is also the equation of motion for a simple spring-mass system under the influence of a forcing function. To add viscous damping to this equation it is necessary to include a term that is proportional to the damping coefficient multiplied by the velocity (first derivative). Thus the well known equation for the motion of a simple spring-mass-dashpot system under the influence of a forcing function is obtained.

$$\ddot{q}_n(t) + 2\eta\ \omega_n\ \dot{q}_n(t) + \omega_n^2\ q_n(t) = f(t)$$

where η is the damping ratio (actual damping coefficient/critical damping coefficient). Since, in this case, the value of η is completely unknown, various different values of η must be tried until the steady state response is detected in the solution.

To observe the effect of damping on this system, therefore, the following system of first order differential equations was programed into the digital computer,

$$\dot{q}_{n_1}(\theta) = q_{n_2}(\theta)$$

$$\dot{q}_{n_2}(\theta) = -\ 2\eta\ [n^4\pi^4 S + n^2\pi^2 Z \cos\theta]^{\frac{1}{2}} q_{n_2}(\theta) -$$

$$[n^4\pi^4 S + n^2\pi^2 Z \cos\theta]\ q_{n_1}(\theta) + f_n(\theta) \qquad (42)$$

where the $f_n(\theta)$ are described above. The Runga-Kutta method of numerical quadratures was also used to solve the set of equations (42).

Partial Differential Equations

In proceeding from one form of the equation of motion for the transverse vibration of the connecting rod of a slider-crank mechanism to another form, simplifying assumptions were required. In order to determine the validity of some of these assumptions the fourth order partial differential equations of motion (20) and (22) were solved. The high speed digital computer IBM 7090 was again used for the computation and a FORTRAN program was compiled using an eleven point finite-difference approach.

By collecting terms and substituting the representative equations for K, expression (18), $\ddot{\alpha}(x,t)$, expression (3a') and $\ddot{\beta}(x,t)$, expression (4a'), equation (20) can be rewritten as

$$\frac{EIL}{M_3} w^{IV}(x,t) + \ddot{w}(x,t) - \Big[(1 + \frac{M_4}{M_3})\, r\, L\, \omega^2 \cos\theta -$$

$$x\, r\, \omega^2 (\cos\theta \cos\varphi - \sin\theta \sin\varphi) - \frac{x^2}{2}\left(\frac{r\omega}{L}\right)^2 \frac{\cos^2\theta}{\cos^3\varphi}\Big] \cdot$$

$$\Big[w''(x,t)\Big] + \Big[r\, \omega^2 (\cos\theta \cos\varphi - \sin\theta \sin\varphi) +$$

$$x\left(\frac{r\,\omega}{L}\right)^2 \frac{\cos^2\theta}{\cos^2\varphi}\Big] w'(x,t) =$$

$$r\, \omega^2 (\cos\theta \sin\varphi + \sin\theta \cos\varphi) -$$

$$x\left\{\frac{r\,\omega^2}{L} \frac{\sin\theta}{\cos\varphi} - \left(\frac{r\,\omega}{L}\right)^2 \frac{\cos^2\theta \sin\varphi}{\cos^3\varphi}\right\}$$

To present the above equation in a more desirable form the

change of variable from t to θ and from x to $\bar{x}$ is required, where $\theta = \omega t$ and, therefore,

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \omega^2 \frac{\partial^2 w(x,\theta)}{\partial \theta^2}$$

and where

$$\bar{x} = \frac{x}{L\sqrt[4]{S}}$$

so that

$$\frac{\partial w}{\partial x} = \frac{1}{L\sqrt[4]{S}}\,\frac{\partial w}{\partial \bar{x}},\quad \frac{\partial^2 w}{\partial x^2} = \frac{1}{L^2\sqrt{S}}\,\frac{\partial^2 w}{\partial \bar{x}^2},\quad \frac{\partial^4 w}{\partial x^4} = \frac{1}{L^4 S}\,\frac{\partial^4 w}{\partial \bar{x}^4}$$

S is the dimensionless parameter defined in equation (39). Performing this operation yields the equation

$$\begin{aligned} & w^{IV}(\bar{x},\theta) + \ddot{w}(\bar{x},\theta) - \left[\frac{Z}{\sqrt{S}}\cos\theta - \frac{\bar{x}}{\sqrt[4]{S}}\left(\frac{r}{L}\right)(\cos\theta\cos\varphi - \sin\theta\sin\varphi)\right. \\ & \quad \left. - \frac{\bar{x}^2}{2}\left(\frac{r}{L}\right)^2\frac{\cos^2\theta}{\cos^2\varphi}\right] w''(\bar{x},\theta) + \left[\frac{r}{\sqrt[4]{S}\,L}(\cos\theta\cos\varphi - \sin\theta\sin\varphi)\right. \\ & \quad \left. + \bar{x}\left(\frac{r}{L}\right)^2\frac{\cos^2\theta}{\cos^2\varphi}\right] w'(\bar{x},\theta) = \\ & \quad r(\cos\theta\sin\varphi + \sin\theta\cos\varphi) - \\ & \quad \bar{x}\sqrt[4]{S}\left\{r\frac{\sin\theta}{\cos\varphi} - \frac{r^2}{L}\frac{\cos^2\theta\sin\varphi}{\cos^3\varphi}\right\} \end{aligned} \tag{43}$$

In this case, the prime notation now denotes differentiation with respect to $\bar{x}$ and the dot notation denotes differentiation with respect to θ. It can be seen that this differential

equation of motion also depends upon the same parameters, S, Z, r and r/L, as the uncoupled equations of motion previously derived. The dependence of this equation on the parameter r can be eliminated by redefining the vibrational amplitude as

$$\bar{w}(\bar{x},\theta) = w(\bar{x},\theta)/r$$

then the three dimensionless parameters S, Z and r/L can be varied to determine their effect on the vibration of the connecting rod.

To complete the statement of the problem, the boundary conditions for the solution of equation (43) written in terms of $\bar{x}$ and θ are

$$w(0,\theta) = 0, \qquad w\left(\frac{1}{\sqrt[4]{S}}, \theta\right) = 0,$$

$$w''(0,\theta) = 0, \qquad w''\left(\frac{1}{\sqrt[4]{S}}, \theta\right) = 0,$$

and the initial conditions are

$$w(\bar{x},0) = 0, \qquad \dot{w}(\bar{x},0) = 0.$$

To simplify the notation of equation (43) let

$$f(\bar{x},\theta) = \left[-\frac{Z}{\sqrt{S}}\cos\theta + \frac{\bar{x}}{\sqrt[4]{S}}\left(\frac{r}{L}\right)(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + \frac{\bar{x}^2}{2}\left(\frac{r}{L}\right)^2\frac{\cos^2\theta}{\cos^3\varphi}\right]$$

$$f'(\bar{x},\theta) = \left[\frac{r}{\sqrt[4]{S}L}(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + \bar{x}\left(\frac{r}{L}\right)^2\frac{\cos^2\theta}{\cos^3\varphi}\right]$$

and

$$g(\bar{x},\theta) = \left[r(\cos\theta\sin\varphi + \sin\theta\cos\varphi) - \bar{x}\sqrt[4]{S}\left\{r\frac{\sin\theta}{\cos\varphi} - \frac{r^2}{L}\frac{\cos^2\theta\sin\varphi}{\cos^3\varphi}\right\}\right].$$

Therefore equation (42) can now be written in the condensed form

$$w^{IV}(\bar{x},\theta) + \ddot{w}(\bar{x},\theta) + \left[f(\bar{x},\theta)\, w'\,(\bar{x},\theta)\right]' = g(\bar{x},\theta) \qquad (44)$$

By arranging the grid and using the difference scheme as shown in Figure 5 the following difference equation was derived[5] to represent equation (44)

$$\begin{aligned} w_{j+1,k+1} + w_{j-1,k+1} - \left(2 + \frac{(\Delta \bar{x})^4}{(\Delta \theta)^2}\right) w_{j,k+1} = \\ 2\left(1 - \frac{(\Delta \bar{x})^4}{(\Delta \theta)^2} - (\Delta \bar{x})^2 f_{j,k}\right) w_{j,k} + \\ \left(-2 + (\Delta \bar{x})^2 f_{j,k} + \frac{(\Delta \bar{x})^3}{2} f'_{j,k}\right) w_{j+1,k} + \\ \left(-2 + (\Delta \bar{x})^2 f_{j,k} - \frac{(\Delta \bar{x})^3}{2} f'_{j,k}\right) w_{j-1,k} + \\ w_{j+2,k} + w_{j-2,k} + \left(2 + \frac{(\Delta \bar{x})^4}{(\Delta \theta)^2}\right) w_{j,k-1} - \\ w_{j+1,k-1} - w_{j-1,k-1} - (\Delta \bar{x})^4 g_{j,k} \end{aligned} \qquad (45)$$

In this difference equation j represents the increment of displacement, $\Delta \bar{x}$, along the connecting rod and varies from 0 to n+1, the total number of displacement intervals; and k represents the increment in angular input, $\Delta \theta$, that varies from time equals 0 to any arbitrary stopping time. The boundary conditions reduce to

$$w_{0,k} = 0, \qquad w_{n+1,k} = 0,$$

$$w_{1,k} = -w_{-1,k}, \qquad w_{n,k} = -w_{n+2,k},$$

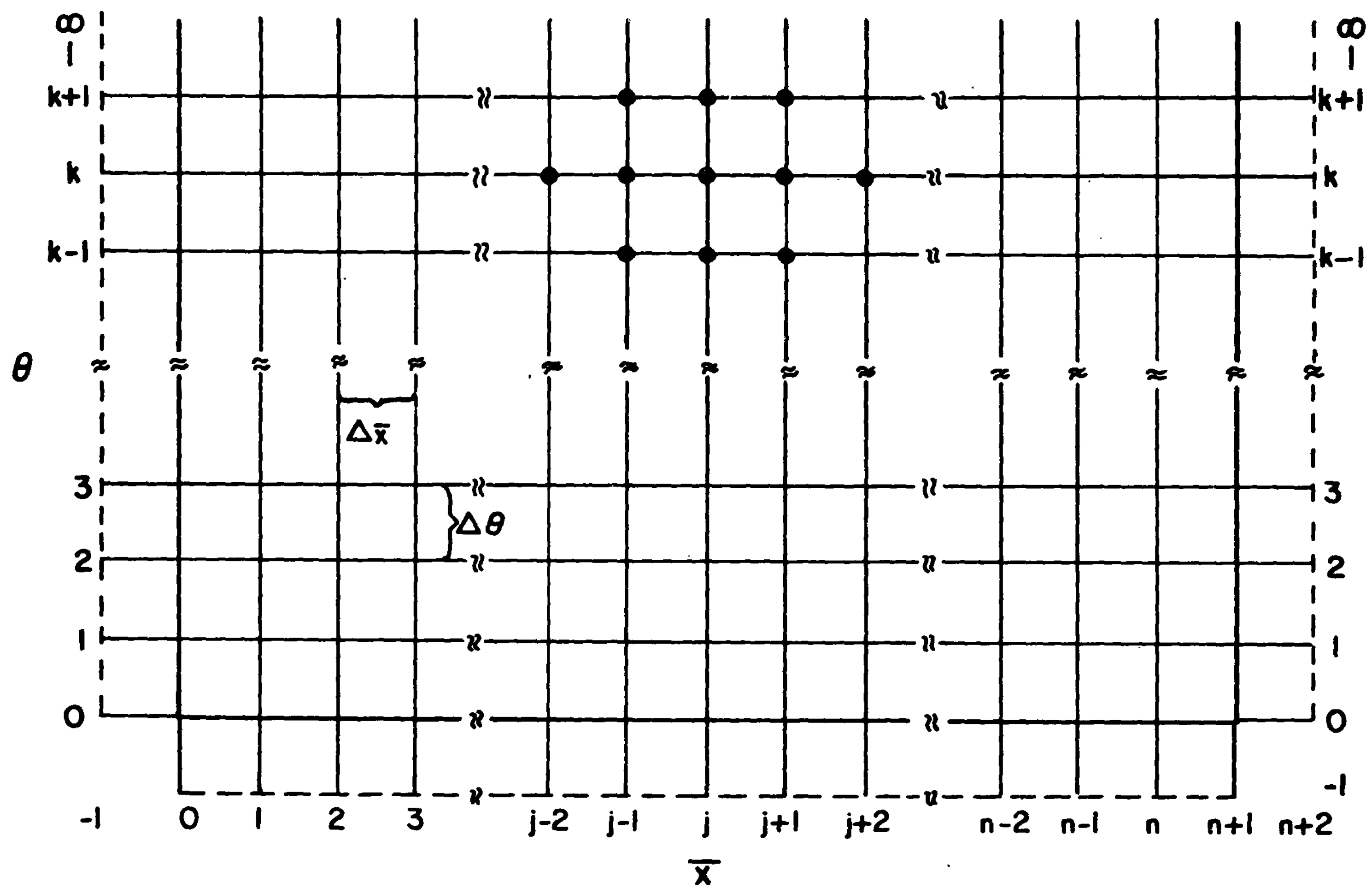

Figure 5. Grid for Finite Difference Solution

and the initial conditions become

$$w_{j,0} = 0, \qquad w_{j,1} = w_{j,-1}.$$

To program this equation into the computer the recursion matrix form of equation (45) is useful. This can be written as

$$\bar{A}\, W^{k+1} = \bar{B}^{k}\, W^{k} - \bar{A}\, W^{k-1} - G^{k} \tag{46}$$

where the matrix $\bar{A}$ is the following (n x n) symmetric, tri -diagonal matrix with constant elements that depend upon the displacement and time increments only

$$\bar{A} = \begin{bmatrix} -\left(2+\frac{(\Delta\,\bar{x})^4}{(\Delta\,\theta)^2}\right) & 1 & 0 & \cdot & \cdot \\ 1 & -\left(2+\frac{(\Delta\,\bar{x})^4}{(\Delta\,\theta)^2}\right) & 1 & \cdot & \cdot \\ 0 & 1 & -\left(2+\frac{(\Delta\,\bar{x})^4}{(\Delta\,\theta)^2}\right) & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \end{bmatrix}$$

Matrix $\bar{B}^k$ is the (n x n) matrix of coefficients

$$\bar{B}^k = \begin{bmatrix} (\gamma_1^k-1) & (\delta_1^k+\epsilon_1^k) & 1 & 0 & 0 & \cdot & \cdot \\ (\delta_2^k-\epsilon_2^k) & \gamma_2^k & (\delta_2^k+\epsilon_2^k) & 1 & 0 & \cdot & \cdot \\ 1 & (\delta_3^k-\epsilon_3^k) & \gamma_3^k & (\delta_3^k+\epsilon_3^k) & 1 & \cdot & \cdot \\ 0 & 1 & (\delta_4^k-\epsilon_4^k) & \gamma_4^k & (\delta_4^k+\epsilon_4^k) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \end{bmatrix}$$

where $\gamma^k = (2 - \dfrac{2(\Delta \bar{x})^4}{(\Delta \theta)^2} - 2(\Delta \bar{x})^2 f_{j,k})$,

$$\delta_j^k = (-2 + (\Delta \bar{x})^2 f_{j,k}),$$

and $\epsilon_j^k = \dfrac{(\Delta \bar{x})^3}{2} f_{j,k}$.

The vector G^k represents the non-homogeneous or right hand side of the equation of motion (43)

$$G^k = (\Delta \bar{x})^4 \lfloor g_{1,k},\ g_{2,k},\ g_{3,k},\ \cdots g_{n,k} \rfloor.$$

It can be seen from the matrix equation (46) that the solution of the difference equation at any given increment, w^{k+1}, of time depends only upon the values of the function at the two preceding time increments, w^k and w^{k-1}, and, therefore, in solving this equation the behavior of the function at successively increasing time increments is determined in a systematic manner.

If the same procedure is used for the finite-difference

solution of the equation of motion (22) as was used on equation (20), then the function notation used in equation (44) reduces to

$$f(\overline{x},\theta) = -\frac{Z}{\sqrt{S}} \cos\theta = f(\theta)$$

which is no longer dependent upon displacement $\overline{x}$ and therefore,

$$f'(\overline{x},\theta) = 0.$$

The function $g(\overline{x},\theta)$ remains the same as in equation (44). Therefore, equation (22) is reduced to the condensed form

$$w^{IV}(\overline{x},\theta) + \ddot{w}(\overline{x},\theta) + f(\theta)\, w''(\overline{x},\theta) = g(\overline{x},\theta). \qquad (47)$$

As a check on the validity of the form of equation (47) the uncoupled equations of motion (37') and (38') can once again be derived from equation (47) by using the technique of normal mode expansion.

If the same finite-difference scheme is used on equation (47) as was used on equation (44), then the form of the recursion matrix formula (46) remains the same except that the $\overline{B}^k$ matrix of coefficients is now only time dependent and no longer depends upon displacement, except for dimension. This matrix becomes the time dependent, symmetric, (n x n) matrix

$$\overline{B}^k = \begin{bmatrix} (\gamma^k-1) & \delta^k & 1 & 0 & 0 & \cdot & \cdot \\ \delta^k & \gamma^k & \delta^k & 1 & 0 & \cdot & \cdot \\ 1 & \delta^k & \gamma^k & \delta^k & 1 & \cdot & \cdot \\ 0 & 1 & \delta^k & \gamma^k & \delta^k & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \end{bmatrix}$$

where γ^k and δ^k are the same as those previously defined except that f_k replaces $f_{j,k}$.

With the substitution of the above matrix for $\overline{B}^k$ into equation (46), this recursion formula can be programed into the computer to obtain the solution to the difference equation representing the equation of motion (22).

SUMMARY AND RESULTS

The difference equations representing the fourth order form (20) and (22) of the equation of motion, and the uncoupled forms (37'), (38') and (42) of the equation of motion were programed and solved on a high speed digital computer using numerical techniques. Discounting diagnostic and de-bugging time, the solution of either difference equation required approximately 0.87 hours of computer time and the solution of the first uncoupled equation required approximately 0.05 hours of computer time. In both cases the selected increment of angular crank input was 0.5° and for the solution of the difference equation 50 displacement increments were used.

The effect of varying the size or number of the $\Delta\,\theta$ or $\Delta\,\bar{x}$ increments on a solution was investigated. It was determined that, in the solution of the uncoupled equations a change from $\Delta\,\theta = 0.5^\circ$ to $\Delta\,\theta = 0.25^\circ$ had little or no effect on the solution to eight decimal places. In the solution of the difference equation, however, an increase in $\Delta\,\theta$ from 0.5° to 5° caused the solution to become unstable and no correlation was achieved. It was also determined that, if the total number of displacement intervals is reduced from 50 to 10, then the solution

of the difference equation approaches the solution of the first uncoupled equation. See Figure 7. No further investigation of this phenomenon was attempted.

Effect of Assuming Variables Separable

To arrive at the uncoupled equations of motion from the simplified, linear partial differential equation of motion a technique based on the method of separation of variables was used (see expression (26)). In previous work in the field of vibrating beams this technique has been successfully used to great advantage; however, for this specific problem this method falls short of giving a complete and accurate description of the solution of the governing partial differential equation because of approximations needed in the development of the uncoupled equations. To compare the solution of the difference equation with the solution of the first uncoupled equation see Figure 6. From the inspection of Figure 6 it can be seen that good correlation between the solutions is achieved during the first and third revolutions of the crank but during the second revolution the two solutions are completely out of phase and the large negative displacement predicted by the uncoupled equation is not predicted by the governing partial differential equation. It can therefore be seen that the uncoupled equation predicts a more vigorous vibration than does the difference

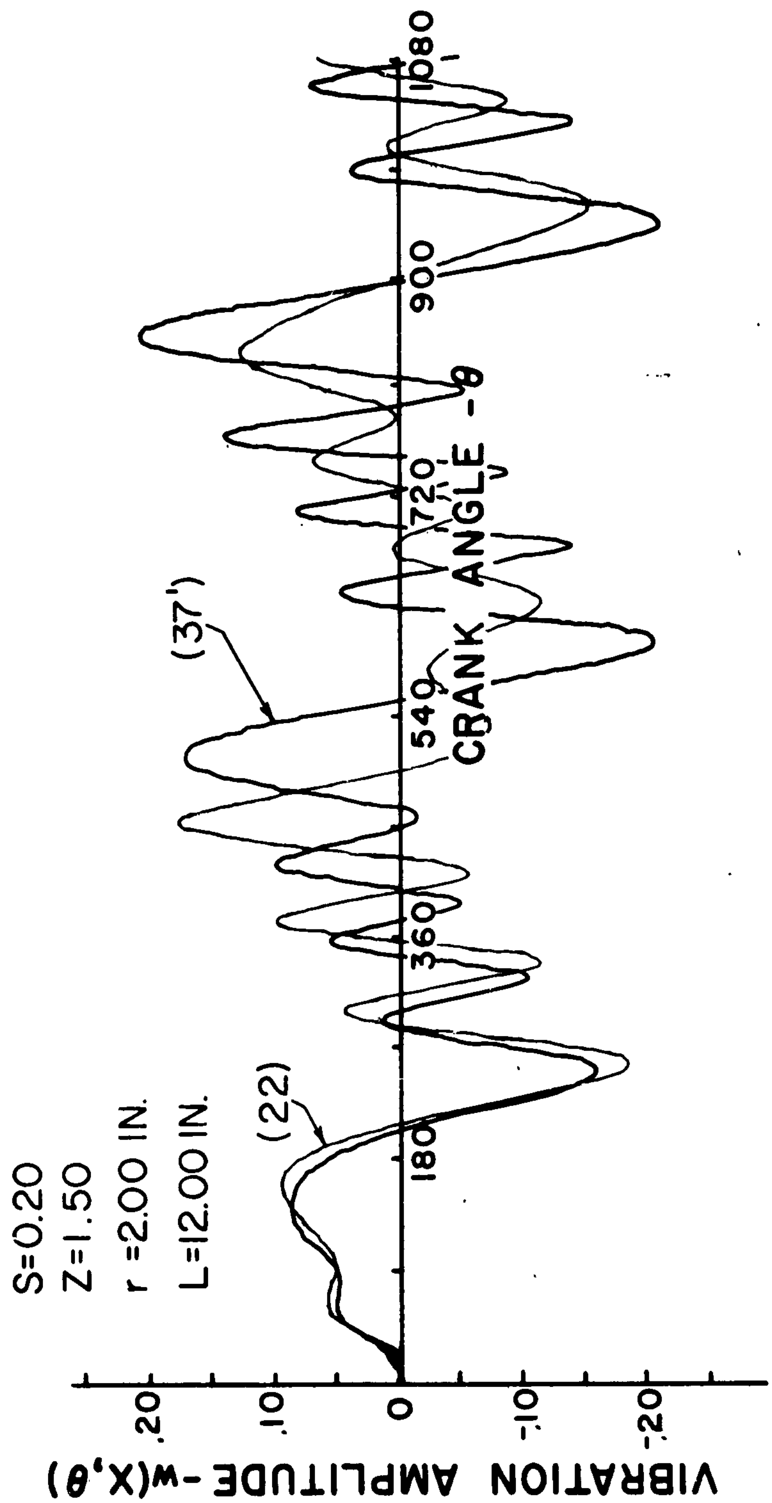

Figure 6. Solutions of First Uncoupled Equation and Simplified Equation (22) 50 Displacement Increments.

equation. The solution of the difference equation, however, exhibits more irregularities and is not quite as repetitive in nature as is the solution of the first uncoupled equation. This would tend to indicate that the uncoupled solution is only a periodic approximation of the difference equation. The addition of more uncoupled solutions, second, third, etc., will have little affect on alleviating this problem of non-correlation.

This indicates that this method of normal mode expansion must be used with caution for this type of problem, and that its validity depends upon the form of the forcing function (right hand side of equation (22)). In this particular case, this technique is not the best because the forcing function is a linear function in displacement and does not lend itself to proper separation. During the process of uncoupling this forcing function was reduced to a function of time only by evaluating the integral on the right hand side of expression (29). In doing this the effect of the x varying function is approximated by its behavior at $x = L/2$ only. Since the right hand side of the equation of motion represents the transverse component of rigid body acceleration for any point on the connecting rod, then this approximation does not seem reasonable. However, with a decrease in the number of displacement increments used in the solution of the difference equation, the resulting solution approaches that

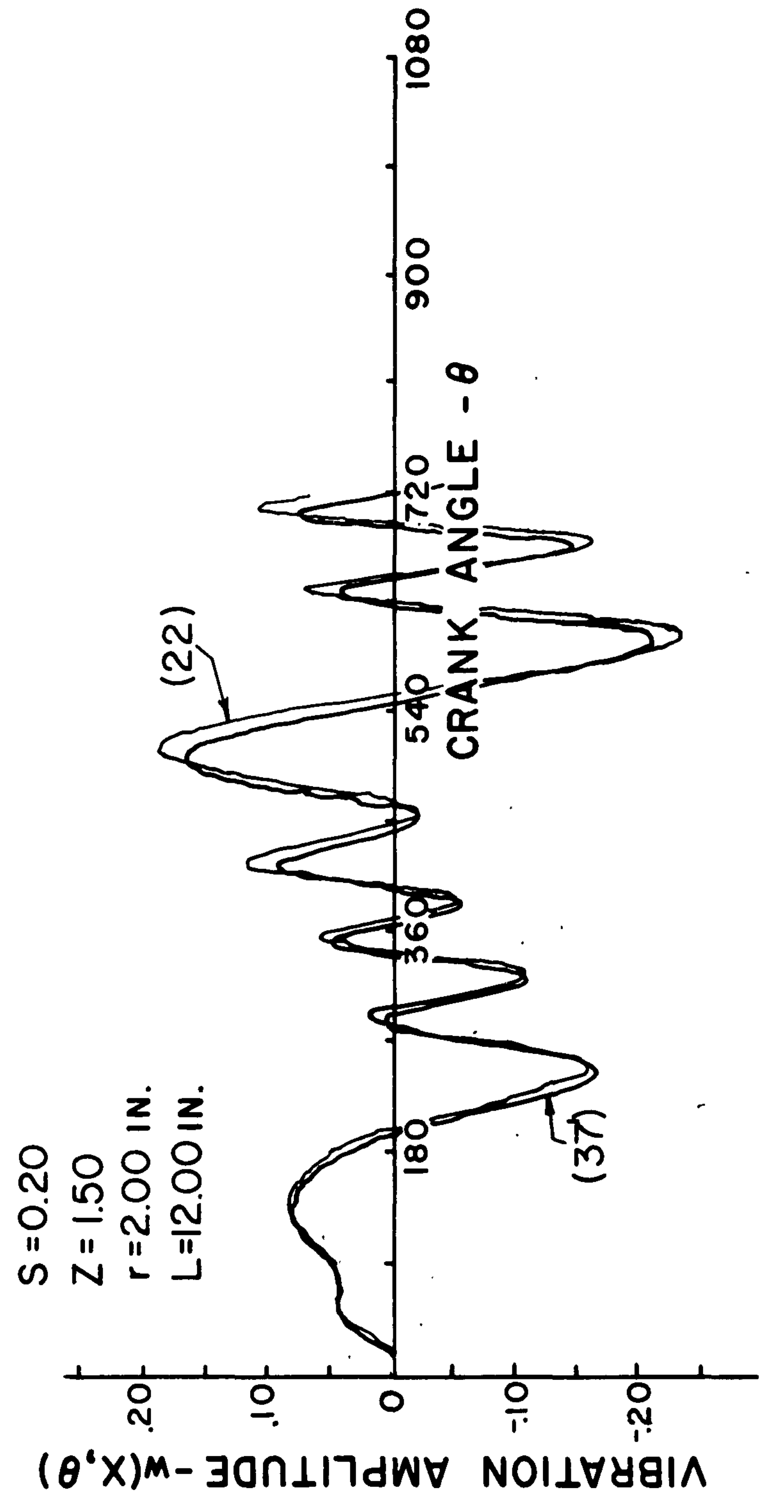

Figure 7. Solutions of First Uncoupled Equation and Simplified Equation (22) 10 Displacement Increments.

arrived at by solving the first uncoupled equation. See Figure 7 for this comparison. This indicates that the solution of the first uncoupled equation does, in part, represent the solution of the partial differential equation and can, therefore, be used to determine the gross effects of varying S or Z.

From Figure 8 it can be seen that the assumption

$$\varphi_n(x) = \sin \frac{n\pi x}{L}$$

is not only valid but that a further assumption is in order. This assumption is that all of the modes can be neglected except the first mode or, in other words, that the solution will take the form

$$w(x,t) = q(t) \sin \frac{\pi x}{L} \qquad (47)$$

rather than be an infinite sum. If this expression is substituted into the equation of motion and if the right hand side of this equation is approximated by the first term of its half range Fourier sine series, then a better approximation of the solution of the difference equation may be obtained. No analytical work was done to substantiate the above statement.

The validity of the assumption in expression (47) can also be substantiated if a comparison between the natural frequencies ω_n and the input crank speed ω is

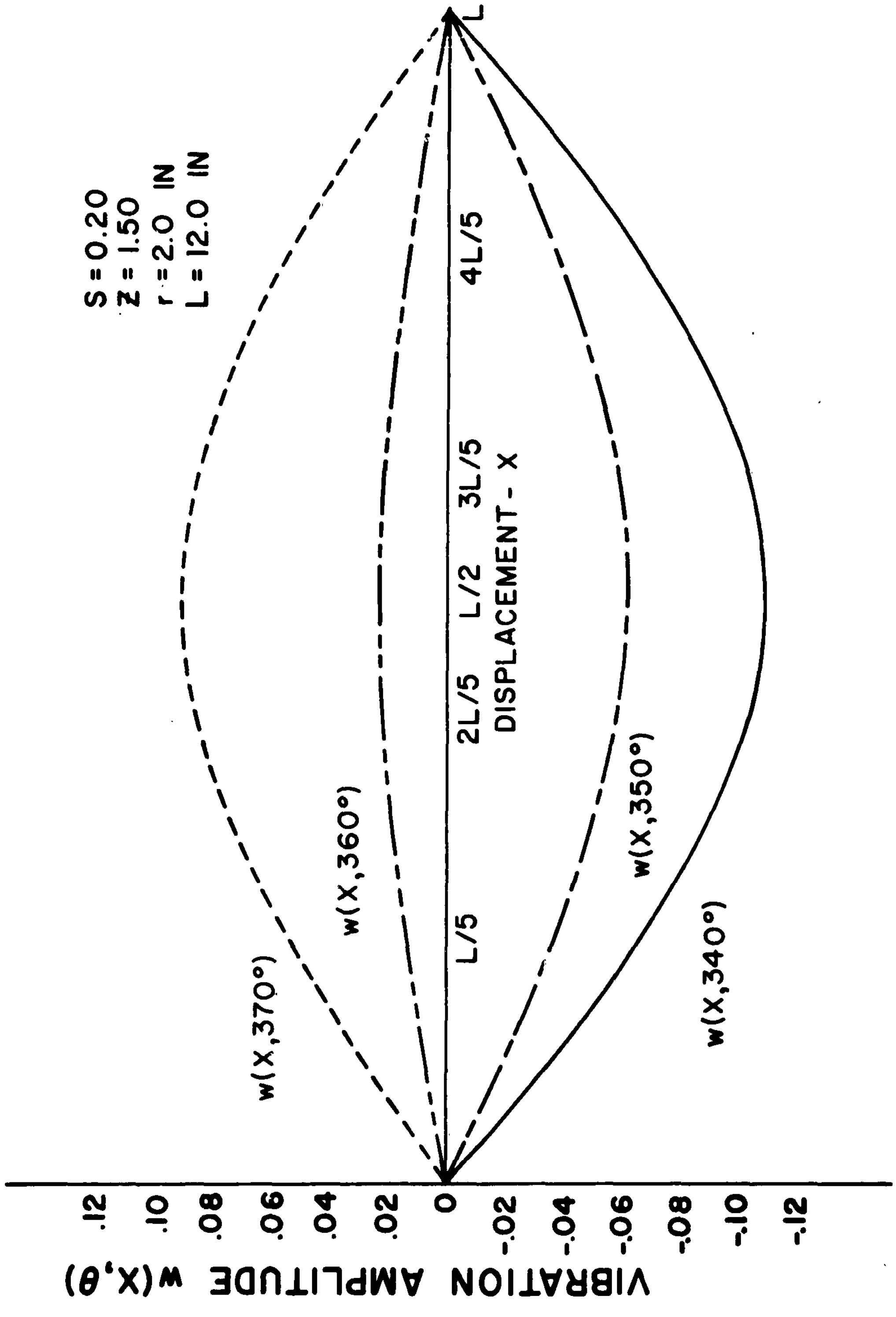

Figure 8. Vibrational Amplitude Distribution Along the Connecting Rod From Solution Of Equation (20).

made. Since ω is the angular frequency of the forcing function a resonant condition will be established for any mode whose natural frequency is close to ω, and little or no detectable excitation will occur for natural frequencies above this value. The first natural frequency ω_1 can vary in value above and below the value of the constant ω depending upon the relative magnitudes of Z and $\pi^2 S$, but the second natural frequency ω_2, which is always much greater than the maximum value of ω_1, will not be excited by ω to as great an extent as ω_1. For the third and higher natural frequencies this is also true because these frequencies increase in proportion to n^4 so that the excitation of the higher modes by ω is negligible. This line of reasoning was also substantiated during the solution of the uncoupled equations of motion (S = 0.2, Z = 1.5, r = 2.0 inches and L = 12.0 inches) when it was calculated that the maximum value of vibrational amplitude of the second mode equation was always less than 10^{-2} times the value of the first mode amplitude and that the maximum value of the vibrational amplitude for the third mode was approximately 10^{-3} times that of the first mode.

Effect of Assuming Small Slope

The assumption of small slope (implies small deflection also) is quite common in vibrational analysis.

Figure 9 is a graphical example of why this is a valid assumption. This figure is a graphical representation of the solution of the finite difference approximations for the equations of motion (20) and (22). The only difference between equations (20) and (22) is that an additional term proportional to the slope was assumed to be zero in the derivation of equation (22).

In order to determine analytically why this assumption is valid first compare the orders of magnitude of the components in the coefficient of w'' in equation (43). It can be seen that, for reasonably large ranges in value of the parameters S and Z, the term

$$\frac{Z}{\sqrt{S}} \cos \theta$$

is the dominant factor. The assumption made above only eliminates the remaining negligible terms. Another term eliminated by this assumption is the entire w' term in equation (43). This elimination, at first, would seem unreasonable except that, in the solution of a partial differential equation, the higher order terms have the greatest effect. Therefore, since the equation in question is a fourth order partial differential equation, the fourth order term will have the greatest effect on determining the solution while the first order term will

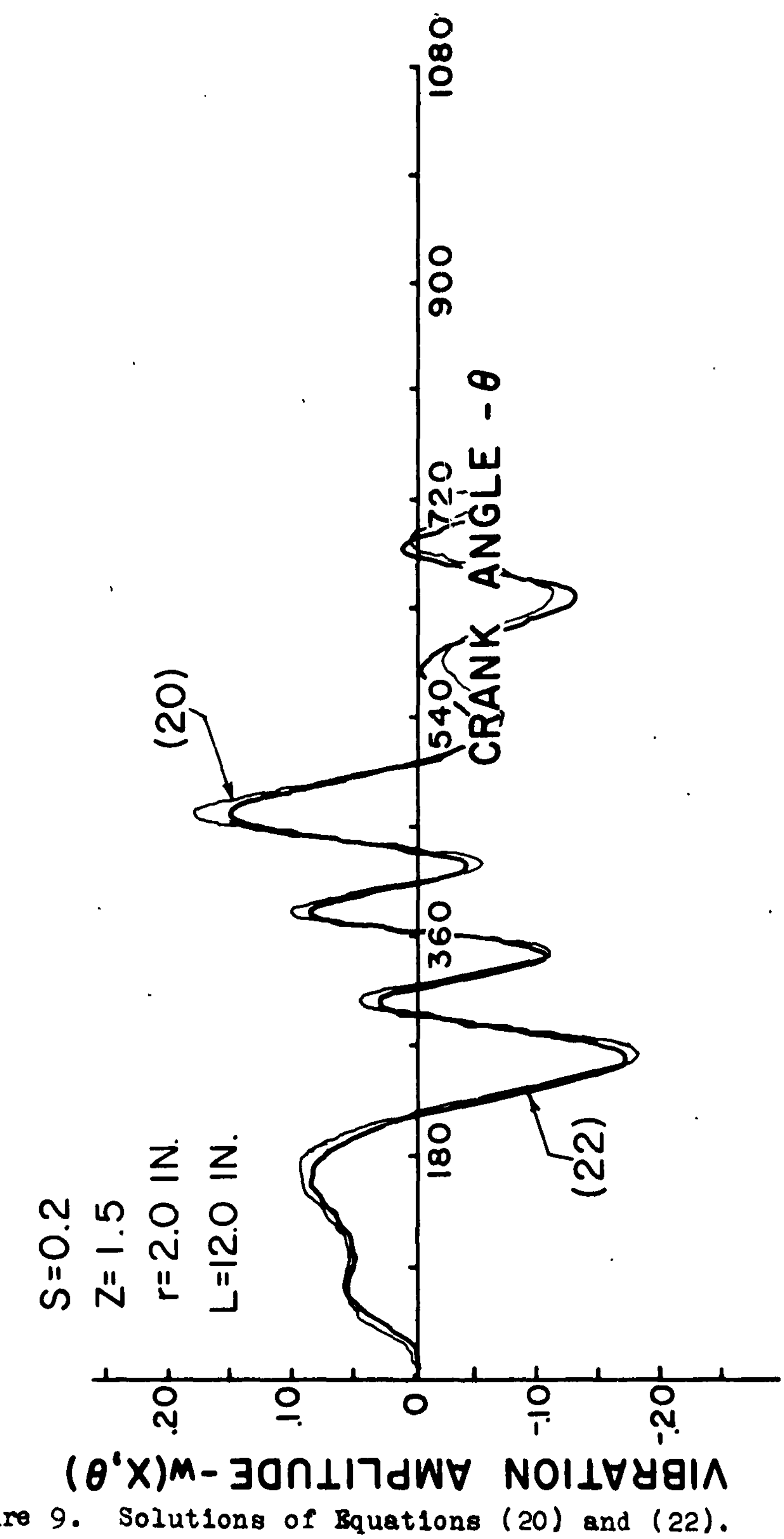

Figure 9. Solutions of Equations (20) and (22).

have the least effect. This, of course, places magnitude restrictions on the coefficient of w' which, in this case, must have been met.

The above is also pointed out in examining the difference equation (45) thereby noting that the only change in the recursion formula (46) in going from the difference equation representing (20) to that representing (22) is in the β^k matrix. Therefore, if $\epsilon_j^k << 1$ and $f_{j,k} \cong f_k$, which is true in this case, then the solutions should be close and, therefore, the assumption valid.

Effect of Damping

The uncoupled equations of motion (37') and (38') were modified to include damping and equation (42) resulted. The value of the damping ratio η was increased in large increments until the effects of the initial conditions on the solution were eliminated. It was determined that a value of $\eta = 0.10$ was sufficient to cause this phenomenon and Figure 10 resulted from the solution of equation (42). Figure 10 is a graphical comparison between a damped and undamped solution on the first uncoupled equation. As would be expected, this figure predicts that the mechanism should behave more sluggishly than is predicted when the natural damping is assumed to be negligible. It can be seen that the peak amplitudes occur at approximately the same time in the cycle but are some-

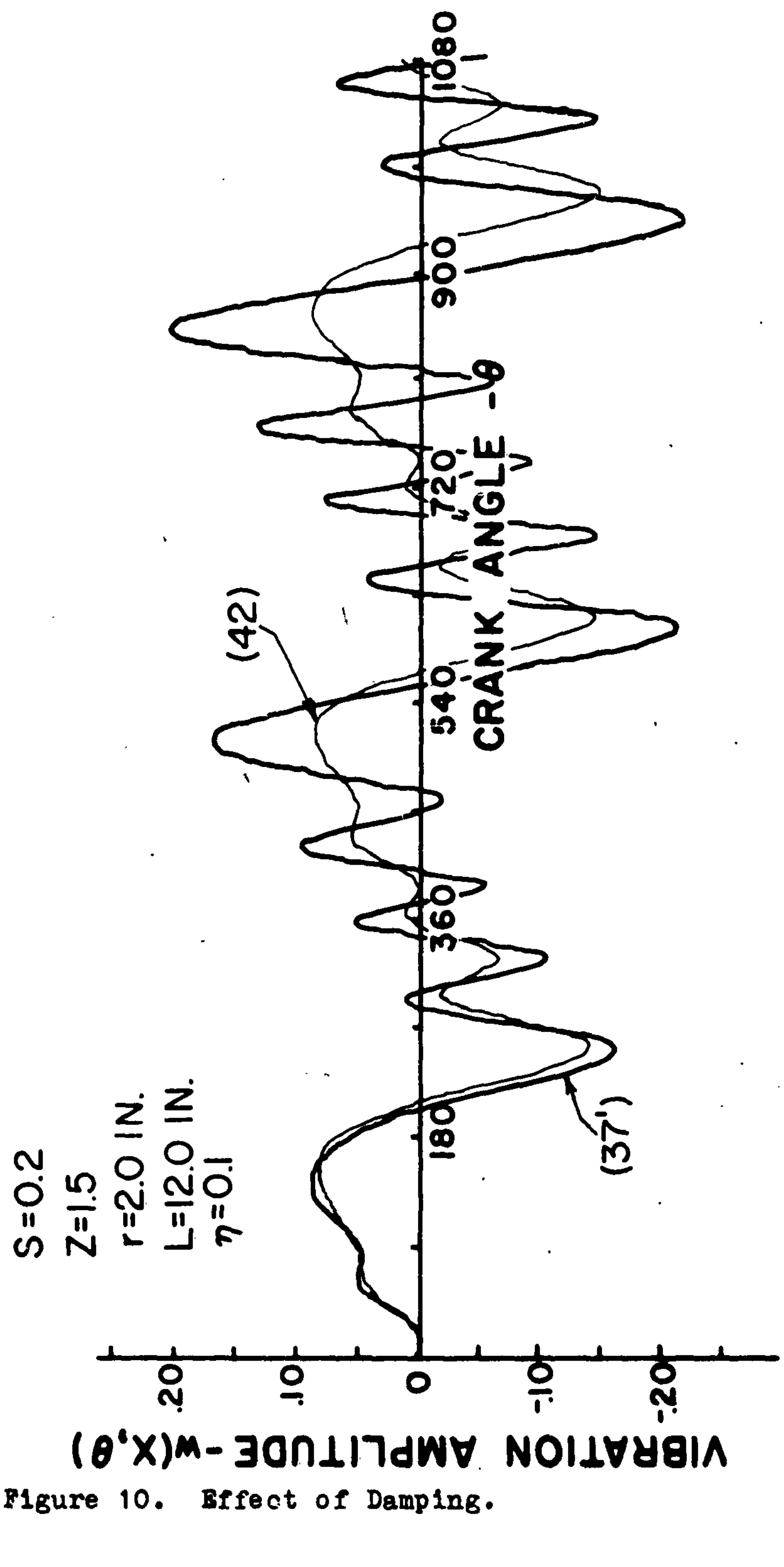

Figure 10. Effect of Damping.

what reduced in magnitude. See Table 1. This fact may indicate that the value $\eta = 0.10$ is too high an estimate for the damping ratio. The important fact to note is that the smaller vibrations between the larger positive and negative amplitude peaks are rapidly damped out whereas the large amplitude peaks are only partially affected.

Predicted Maximum Variations in Piston Displacement

Table 1 presents the maximum variations in piston displacement at the four predominant vibrational peaks that occur during the first two revolutions of the crank. The amplitude, w, and the crank angle, θ, were taken from the data represented in Figures 6, 9 and 10, and Appendix C contains the derivation for the expressions representing ΔL and ΔX_B.

It can be noted that, in the case of Table 1, the deflection, w, is considerably larger than the piston variation, ΔX_B. This is due to the fact that ΔL is proportional to the square of the amplitude, a small number, and inversely proportional to the length of the connecting rod.

For the first two amplitude peaks all of the solutions itemized in Table 1 predict comparable variations at the same time in the cycle. During the second cycle divergence in both piston variation and time of occurrence becomes pronounced.

Table 1. Predicted Variation in Piston Displacement
(r = 2 in, L = 12 in, S = 0.2, Z = 1.5)

Main Peak	Equation	θ	W Amplitude Inches	ΔL Shortening Inches	ΔX_B Inches
1	(20)	155°	.0904	.00168	.00167
	(22)	160°	.0981	.00198	.001968
	(37')	135°	.0908	.001698	.001688
	(42)	150°	.0889	.001625	.001615
2	(20)	250°	-.167	.00573	.00565
	(22)	255°	-.180	.00666	.00657
	(37')	245°	-.157	.00507	.005
	(42)	240°	-.141	.00408	.00402
3	(20)	455°	.158	.00507	.00493
	(22)	460°	.173	.00615	.00597
	(37')	505°	.178	.00651	.00646
	(42)	510°	.0889	.001625	.001615
4	(20)	630°	-.125	.00321	.00312
	(22)	635°	-.1097	.002475	.0024
	(37')	595°	-.207	.0088	.00867
	(42)	600°	-.141	.00408	.00402

Discussion of the Parameters

In order to discuss the possible effect that the dimensionless parameters $\bar{S}$ and Z will have on the vibrational characteristics of a mechanism a sample slider-crank mechanism will be analyzed.

Equations (39) and (40) are expressions representing S and Z respectively. If a value of Z = 1.0 is chosen for the physical parameter and if the linkage properties are reasonably chosen then the following quantities can be

specified

$E = 30 \times 10^6$ lb/in^2, $\qquad r = 1$ in,

$I = 0.0208$ in^4 ($1 \times \frac{1}{4}$ cross-section), $\qquad L = 6$ in,

$M_3 = 0.0022$ lb-sec^2/in, $\qquad M_4 = 0.011$ lb-sec^2/in.

The S, or speed, parameter cannot be specified at this time because the operating input speed ω of the mechanism has not been specified. These quantities will be used throughout this subsection for illustration purposes.

In the uncoupled equations of motion (37') or (38') the coefficient of $q_n(\theta)$ represents the natural frequencies of the modes of vibration. Written in terms of the dimensionless parameters this coefficient becomes

$$\left[\frac{\omega_n}{\omega}\right]^2 = [\pi^4 n^4 S + \pi^2 n^2 Z \cos \theta] \tag{48}$$

Since $0 \leq \theta \leq \infty$, the second term in this expression will become negative when the angle θ is in the second or third quadrant and $\cos \theta = -1.0$ for θ equaling π or any odd multiple of π.

It can therefore be shown that the uncoupled equations of motion no longer predict the vibratory motion of the connecting rod unless

$$Z \leq \pi^2 S \tag{49}$$

for any given situation. If Z exceeds this limit then the first natural frequency, as calculated from expression

(48), will be imaginary, a situation that is contrary to physical reality.

It is interesting to note that the criteria stated in equation (49) corresponds to the situation where the dynamic axial load approaches the Euler criteria for static buckling of long, slender columns. That is

$$K \cos \theta = P_{cr}$$

where

$$P_{cr} = \frac{\pi^2 EI}{L^2} .$$

Although, in general, the connecting rod will not have a sufficiently low slenderness ratio to allow the use of the Euler buckling criteria directly, this condition is a guide and loadings above P_{cr} should be avoided by the designer.

S and Z are the parameters that govern the behavior of the equation of motion. However, since the equation of motion has not been completely nondimensionalized, two other parameters of the mechanism must be specified in order to attempt a solution; these are the crank length, r, and the connecting rod length, L. Since the parameters S and Z are important in analyzing the vibratory properties of a slider-crank

mechanism, a discussion of the effect of mechanism properties on these parameters is necessary.

First consider only the dimensionless ratio r/L. If r/L remains constant then Z will remain constant; however, this can be true even though r and L both vary in a manner consistent with the constant r/L ratio. If r increases then L must increase proportionately. This increase in L will cause a large decrease in S, since S is inversely proportional to L^3. This has the effect of making $\pi^2 S$ approach Z, and, as will be shown in detail later, as these two quantities become equal in value the vibrational oscillations of the connecting rod become greater in both number and amplitude. This has a very decided effect if Z is close to $\pi^2 S$ before L is increased because a slight change in S in this range has a very marked influence on the amplitude of vibration. The corresponding increase in r also causes an increase in the non-homogeneous part of the equation of motion; however, the effect of increasing L is much more pronounced than is the proportionate increase in r. The converse of this statement is also true.

Next consider the mass ratio M_4/M_3. If this ratio remains constant then Z will remain constant, but both M_4 and M_3 can vary as long as the specified ratio is maintained. If M_3 increases then S decreases, since

S is inversely proportional to M_3, and $\pi^2 S$ approaches Z. This condition will cause an increase in vibrational activity. However, since M_3 has less of an effect on S than does L, the vibrational effect mentioned in the above paragraph will be much more pronounced.

The modulus of elasticity, E, depends only upon the material used in the connecting rod, and the moment of inertia, I, depends upon the cross-sectional area of the connecting rod. Although the freedom with which E can be varied is limited and the variation of I is somewhat limited by the mass M_3, these two parameters can be varied independently of Z. Increasing the EI combination increases S and, therefore, decreases the vibrational oscillations. The converse of this statement is also true.

The only remaining quantity that has an effect on the relative value of the dimensionless parameters S and Z is the angular input speed of the crank, ω. The speed parameter S (This parameter was given this name because it is an effective measure of the angular velocity of the crank.) is inversely proportional to ω^2. It must be noted that ω is the only term in S that can be varied over a wide range of values without affecting the remainder of the mechanism parameters.

As would be expected when $\omega = 0$ the speed parameter $S = \infty$. This in turn predicts, through the equation of

motion, that no vibration is present in the connecting rod. As S decreases (or ω increases) the inequality stated in expression (49) approaches the limiting case

$$Z = \pi^2 S$$

The value of ω that gives this equality, for a specified mechanism, is the predicted maximum value for the input speed of the crank. As an example, using the quantities given earlier

$$\omega_{max} = \frac{\pi^2 EI}{M_3 L^3} = 3600 \text{ rad/sec} = 34{,}300 \text{ RPM.}$$

To further illustrate the effect of increasing r and L while keeping the ratio r/L constant, let r = 2 in. and L = 12 in. Performing the calculations for ω_{max} yields

$$\omega_{max} = 1273 \text{ rad/sec} = 12{,}150 \text{ RPM.}$$

It can be seen that the maximum allowable angular input is greatly reduced by this increase in L.

However, if M_3 is doubled while the mass ratio is held constant (The same would be true if E or I would be decreased by one half.) then this calculation yields

$$\omega_{max} = 2550 \text{ rad/sec} = 24{,}400 \text{ RPM.}$$

The effect of this change on the maximum permissible running speed is much less than that recorded from the

change in connecting rod length.

The effects of increasing the angular speed ω on the predicted vibrational characteristics of a mechanism are recorded in Figure 11. The data for the figure were generated by solving the first uncoupled equation of motion with $r = 2$ in., $L = 12$ in., $Z = 1.0$ and $0.13 \leq S \leq 1.0$. Using the necessary quantities from those given at the beginning of this subsection the parameters can be translated into representing the following variations in speed

$$\omega = \sqrt{\frac{EI}{SM_3 L^3}},$$

therefore

$$\begin{aligned} \omega = 405 \text{ rad/sec} &\sim S = 1.0, \\ 574 \text{ rad/sec} &\sim 0.5, \\ 906 \text{ rad/sec} &\sim 0.2, \\ 1047 \text{ rad/sec} &\sim 0.15. \\ 1123 \text{ rad/sec} &\sim 0.13, \end{aligned}$$

It can therefore be seen that for speeds that are slow in comparison to the maximum predicted speed, small, smooth vibrations are recorded with amplitude reversals occurring only every 180°. As the speed is increased the amplitude of vibration increases but the reversals still occur only every 180°. Increasing the speed further, $0.15 \leq S \leq 0.2$, finally causes the reversals to become more numerous. At this speed, and above, the vibrations increase in both amplitude and number at an accelerated rate until the equation of motion becomes unstable and large unbounded vibrations are predicted.

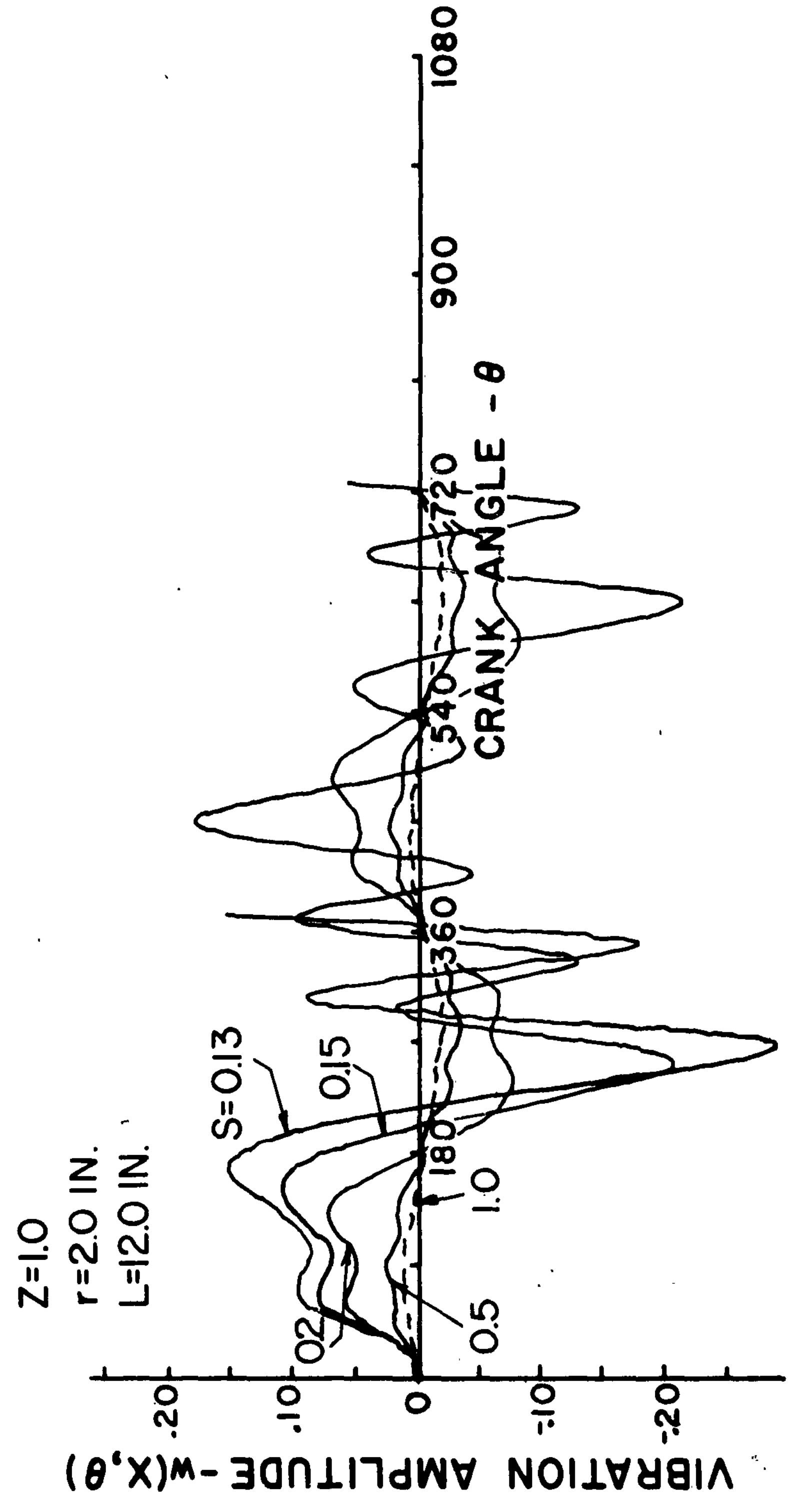

Figure 11. Solution of First Uncoupled Equation for Different Crank Speeds.

Although not recorded on Figure 11 the first uncoupled equation was also solved for $S = 0.11$, a value that barely exceeds the minimum permissible value of S. The trend recorded in Figure 11 was continued and a peak amplitude of approximately 0.9 was recorded at $\theta = 560°$; this represents a piston variation of approximately 0.166 inches. This solution was not included in Figure 11 because unstable predictions were recorded during the third revolution of the crank.

CONCLUSIONS

Equations of motion for the transverse vibrational characteristics of a slider-crank mechanism have been derived and solutions of three different forms of these equations have been obtained for a mechanism with a crank length of 2 inches. The data from these solutions are presented in Figures 6, 8, and 9. The primary objectives set forth at the beginning of this study have been achieved. However, due to the required extended use of computer facilities, the effect of extensive mechanism parameter variation was not obtained.

The uncoupling procedure used during this investigation is not entirely adequate in that it only results in a periodic approximation of the solution to the more complex fourth order form of the equation of motion. However, since the uncoupled equations are sufficient to predict the gross vibrational results, and since these equations predict vibrational activity greater than that predicted by the linear fourth order form of the equation of motion, the uncoupled equations can be used for design purposes. In light of the results recorded in Figure 8 a better approximation of the solution may be obtained by assuming that the amplitudes of all of the modes of vibration are negligible except the first

mode. The mode represented by the first natural frequency is the predominant mode because the value of the frequency of the forcing function is close to the value of the first natural frequency. The second and higher natural frequencies are always much greater than the first natural frequency, therefore, excitation of these modes is much less.

If the restrictions placed on the analysis by the loading approximations derived in Appendix B are too restrictive, then the solution of equation (16') can be undertaken. Obtaining this solution would follow the same procedure as obtaining the solution of equation (20). The only difference will occur in the complexity of the $\overline{B}^k$ matrix in the recursion formula.

Figure 9 shows excellent correlation between a more exact and a simplified version of the linear fourth order equation of motion. Appropriate simplifying assumptions can, therefore, be made without destroying the validity of the resulting solution. However, assumptions should, if possible, be investigated to assure their appropriateness.

As pointed out in Figure 10 a relatively small amount of damping does not affect the major vibrational amplitude peaks to any great degree. Damping does, however, markedly affect the number of complete amplitude reversals so that minor vibrational amplitude peaks are rapidly damped out.

In evaluating the effects of parameters on the vibrational characteristics of the mechanism it is discovered

that the lengths r and L, especially L, have a far greater effect on the response of the system than do other physical parameters such as type of material, cross-section or mass. Increasing L causes a marked increase in vibrational activity and also greatly reduces the maximum operating input speed.

To completely evaluate the effect of parameter variation on the transverse vibration of the connecting rod, equations (37'), (38') and (43) should be non-dimensionalized as suggested and solved for various values of S, Z and r/L.

From this investigation the designer is give one more tool with which to work. Noting that the vibrational amplitude increases as $\pi^2 S$ approaches Z, if accuracy of piston displacement is required then the designer must avoid the situation where Z is approximately equal to $\pi^2 S$. To do this he has at his disposal both the variation of any physical linkage parameter and the variation of the crank speed. Based on the results in Figure 11, if $\pi^2 S$ is approximately twice the value of Z then the predicted transverse vibrations rapidly become negligible.

Certain assumptions were made during this vibrational investigation to arrive at an equation of motion. The effect of the majority of these assumptions cannot be adequately evaluated analytically. Therefore, this analysis should be supplemented with an experimental effort.

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APPENDIX

APPENDIX A

Rigid Body Acceleration Analysis of a Slider-Crank Mechanism

In Figure 1 the length of the line segment $\overline{AC} = r \sin \theta$; the length of line segment $\overline{OC} = r \cos \theta$, and the length of the line segment $\overline{CB} = (L^2 - r^2 \sin^2 \theta)^{\frac{1}{2}}$. The well known equation relating the piston displacement to the angular input is, therefore, derived by noting that

$$X_B = \overline{OC} + \overline{CB} = r \cos \theta + L(1 - r^2/L^2 \sin^2 \theta)^{\frac{1}{2}}. \qquad (1)$$

By making use of the binomial expansion and by restricting the investigation to small r/L ratios ($r/L < \frac{1}{4}$), the radical term can be replaced by

$$(1 - r^2/L^2 \sin^2\theta)^{\frac{1}{2}} \cong 1 - \tfrac{1}{2} r^2/L^2 \sin^2 \theta.$$

Therefore,

$$X_B \cong r \cos \theta + L(1 - \tfrac{1}{2} r^2/L^2 \sin^2 \theta). \qquad (1a)$$

The approximate acceleration of the piston, B, is found by differentiating equation (1a) twice with respect to time.

$$\dot{X}_B = - r \dot{\theta} \sin \theta - r^2/2L \dot{\theta} \sin 2\theta,$$

$$\ddot{X}_B = -(r \cos \theta + r^2/L \cos 2\theta)\dot{\theta}^2 - (r \sin \theta + r^2/2L \sin 2\theta)\ddot{\theta}. \qquad (2a)$$

The x and y components of the rigid body acceleration of a general point ξ on the connecting rod are determined by summing the appropriate components of the acceleration of point A and of the relative acceleration of ξ with respect to A,

$$X_A = r\cos\theta, \qquad Y_A = r\sin\theta,$$

$$\dot{X}_A = -r\dot{\theta}\sin\theta, \qquad \dot{Y}_A = r\dot{\theta}\cos\theta,$$

$$\ddot{X}_A = -r\dot{\theta}\cos\theta - r\ddot{\theta}\sin\theta, \quad \ddot{Y}_A = -r\dot{\theta}\sin\theta + r\ddot{\theta}\cos\theta.$$

Using an orthogonal coordinate transformation from the (X,Y) system to the (x,y) system gives the components of the acceleration of point A in the x and y directions

$$\ddot{x}_A = \ddot{X}_A\cos\varphi - \ddot{Y}_A\sin\varphi.$$

$$\ddot{y}_A = \ddot{X}_A\sin\varphi + \ddot{Y}_A\cos\varphi.$$

The acceleration of some arbitrary point ξ on the connecting rod can now be written as

$$\begin{aligned}\ddot{\alpha}(\xi,t) &= \ddot{x}_\xi = \ddot{x}_A + A^n_{\xi A}\\ &= \ddot{X}_A\cos\varphi - \ddot{Y}_A\sin\varphi - \xi\dot{\varphi}^2\\ &= -r\dot{\theta}^2(\cos\theta\cos\varphi - \sin\theta\sin\varphi)\\ &\quad -r\ddot{\theta}(\sin\theta\cos\varphi + \cos\theta\sin\varphi) - \xi\dot{\varphi}^2, \qquad (3a)\end{aligned}$$

$$\begin{aligned}\ddot{\beta}(\xi,t) &= \ddot{y}_\xi = \ddot{y}_A + A^t_{\xi A}\\ &= \ddot{X}_A\sin\varphi + \ddot{Y}_A\cos\varphi + \xi\ddot{\varphi}\\ &= -r\dot{\theta}(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\\ &\quad -r\ddot{\theta}(\sin\theta\sin\varphi - \cos\theta\cos\varphi) + \xi\ddot{\varphi}. \qquad (4a)\end{aligned}$$

In equation (4a) and (3a) $A^n_{\xi A}$ represents the normal component of the acceleration of ξ relative to A and $A^t_{\xi A}$ represents the tangential component of the acceleration of ξ relative to A. (Positive $\ddot{\varphi}$ must be counterclockwise about the point A for this to hold.)

From the coordinate transformation it can also be shown that

$$\ddot{\beta}(L,t) = \ddot{X}_B \sin \varphi$$
$$\ddot{\alpha}(L,t) = \ddot{X}_B \cos \varphi.$$

The above equations have served as a check on the derivation.

Two relationships that can be used to express φ as a function of θ are also available from the geometry of Figure 1

$$\sin \varphi = r/L \sin \theta, \tag{5a}$$
$$\cos \varphi = (1 - r^2/L^2 \sin^2 \theta)^{\frac{1}{2}}. \tag{6a}$$

Equation (5a) can be used to determine $\dot{\varphi}$ and $\ddot{\varphi}$ as functions of the angular input θ. The first derivative of (5a) with respect to time yields

$$\dot{\varphi} \cos \varphi = \frac{r\dot{\theta}}{L} \cos \theta, \tag{7a}$$

or $$\dot{\varphi} = \frac{r\dot{\theta}}{L} \frac{\cos \theta}{\cos \varphi}.$$

Care must be taken to assure that the sign of this function representing $\dot{\varphi}$ agrees with the assumed convention of counterclockwise rotation about A. It can be seen from a velocity polygon representing the velocities of the mechanism in the

position shown in Figure 1 that the relative velocity of B with respect to A will be directed in the negative y direction. Therefore, to keep $\dot{\varphi}$ within the chosen sign convention

$$\dot{\varphi} = -\frac{r\dot{\theta}}{L}\,\frac{\cos\theta}{\cos\varphi}. \tag{8a}$$

Differentiating equation (7a) with respect to time yields the angular acceleration of the connecting rod

$$-\sin\varphi\,\dot{\varphi}^2 + \cos\varphi\,\ddot{\varphi} = \frac{r}{L}(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta}),$$

$$\ddot{\varphi} = \frac{r}{L\cos\varphi}(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta}) + \frac{\sin\varphi}{\cos\varphi}\,\dot{\varphi}^2. \tag{9a}$$

Substituting equation (8a) into equation (9a) gives the expression

$$\ddot{\varphi} = \frac{r}{L\cos\varphi}(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta}) + \left(\frac{r\dot{\theta}}{L}\right)^2 \frac{\cos^2\theta\sin\varphi}{\cos^3\varphi}.$$

It can be seen from an acceleration polygon representing the accelerations of the mechanism in the position shown in Figure 1 that the tangential component of the acceleration of B relative to A is directed in the negative y direction which means that $\ddot{\varphi}$ is clockwise about A. Therefore, the keep $\ddot{\varphi}$ within the chosen convention of positive being counterclockwise a negative sign must be included. Therefore,

$$\ddot{\varphi} = \frac{r}{L\cos\varphi}(\sin\theta\,\dot{\theta}^2 - \cos\theta\,\ddot{\theta}) - \left(\frac{r\dot{\theta}}{L}\right)^2 \frac{\cos^2\theta\sin\varphi}{\cos^3\varphi}. \tag{10a}$$

If the angular input $\theta(t)$ is a linear function of time, $\theta = \omega t$, then the basic equations given above reduce to

$$\ddot{X}_B = -r\,\omega^2(\cos\theta + \frac{r}{L}\cos 2\theta), \tag{2a'}$$

$$\ddot{\alpha}(\xi,t) = -r\,\omega^2(\cos\theta\cos\varphi - \sin\theta\sin\varphi) - \xi\,\dot{\varphi}^2, \tag{3a'}$$

$$\ddot{\beta}(\xi,t) = -r\,\omega^2(\cos\theta\sin\varphi + \sin\theta\cos\varphi) + \xi\,\ddot{\varphi}, \tag{4a'}$$

$$\dot{\varphi} = -\frac{r\omega}{L}\frac{\cos\theta}{\cos\varphi} \tag{8a'}$$

$$\ddot{\varphi} = \frac{r\omega^2}{L}\frac{\sin\theta}{\cos\varphi} - (\frac{r\,\omega}{L})^2\,\frac{\cos^2\theta\sin\varphi}{\cos^3\varphi} \tag{10a'}$$

It can readily be seen that $\ddot{\alpha}$ and $\ddot{\beta}$ are linear functions in displacement and for simplicity can be written as

$$\ddot{\alpha}(\xi,t) = g_1(t) + \xi g_2(t) \tag{3a''}$$

$$\ddot{\beta}(\xi,t) = g_3(t) + \xi g_4(t) \tag{4a''}$$

where the functions of time, or θ, are described in equations (3a'), (4a'), (8a') and (10a') above.

APPENDIX B

Rigid Body Force Analysis

If the assumption is made that the vibration of the connecting rod does not affect the pin reactions at A and B, then R_x and R_y become known functions of time, $\theta(t)$, that can be determined directly from equations (13) and (11) respectively. These functions, however, are relatively complex in nature and do not lend themselves to simple mathematical formulation. With the hope of determining a simple approximation for R_x and R_y that can be used in equation (14) thereby eliminating the non-linearities of the final equation (16), a FORTRAN program for the IBM 7090 was compiled. This program was used to calculate the components of the A and B pin reactions perpendicular and parallel to the connecting rod. The slider-crank mechanism analyzed was assumed to have a constant input crank speed, ω; the connecting rod was assumed to be a long, thin rod, and the external piston force, P(t), was assumed to be zero. The dimensionless ratios r/L and M_3/M_4 were varied to determine the effect that these ratios had on the reactions.

The non-dimensionalized results of this program are graphically illustrated in Figures 12, 13, 14 and 15. The

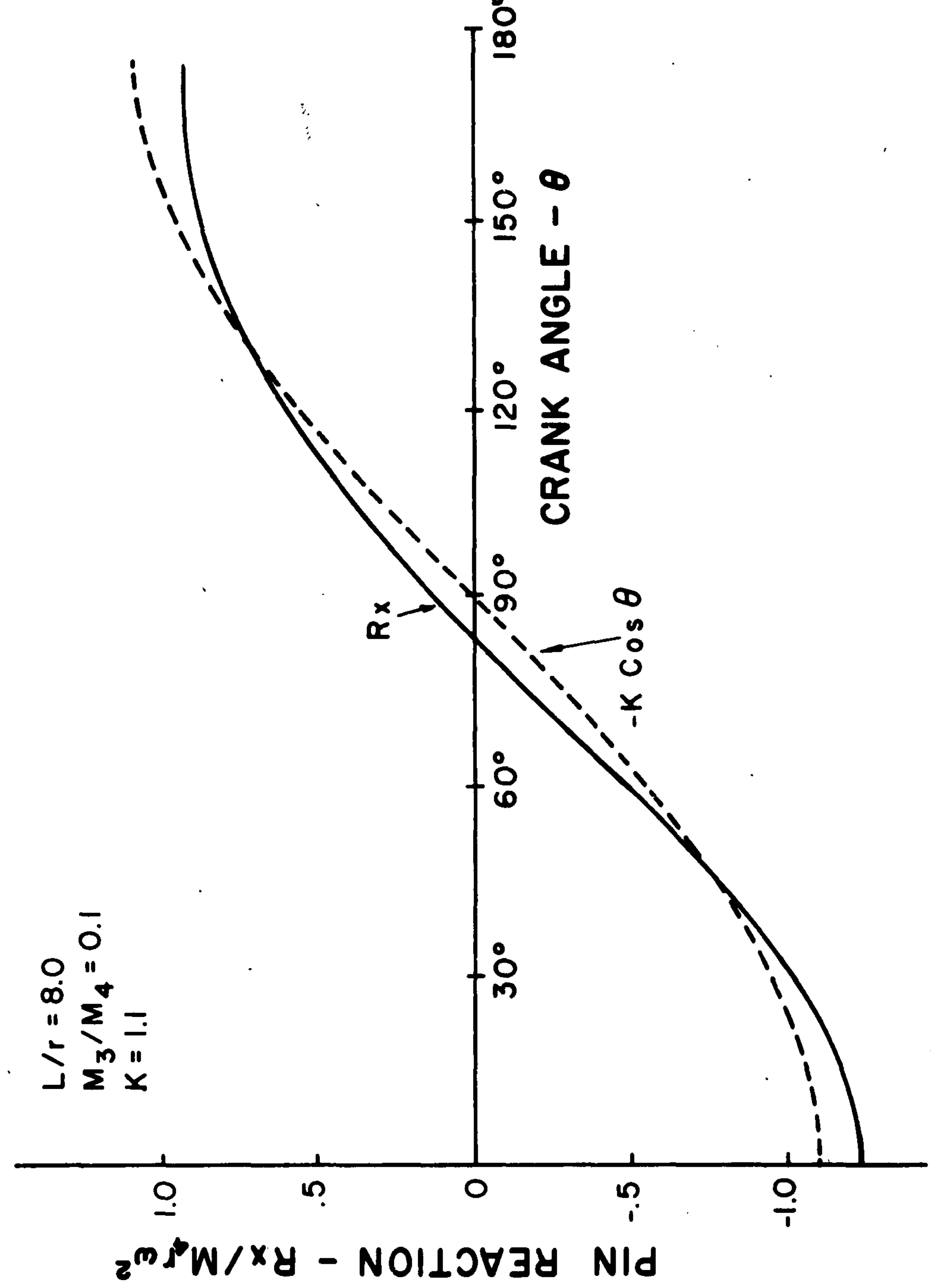


Figure 12. Pin Reaction R_x Compared with - K cos θ

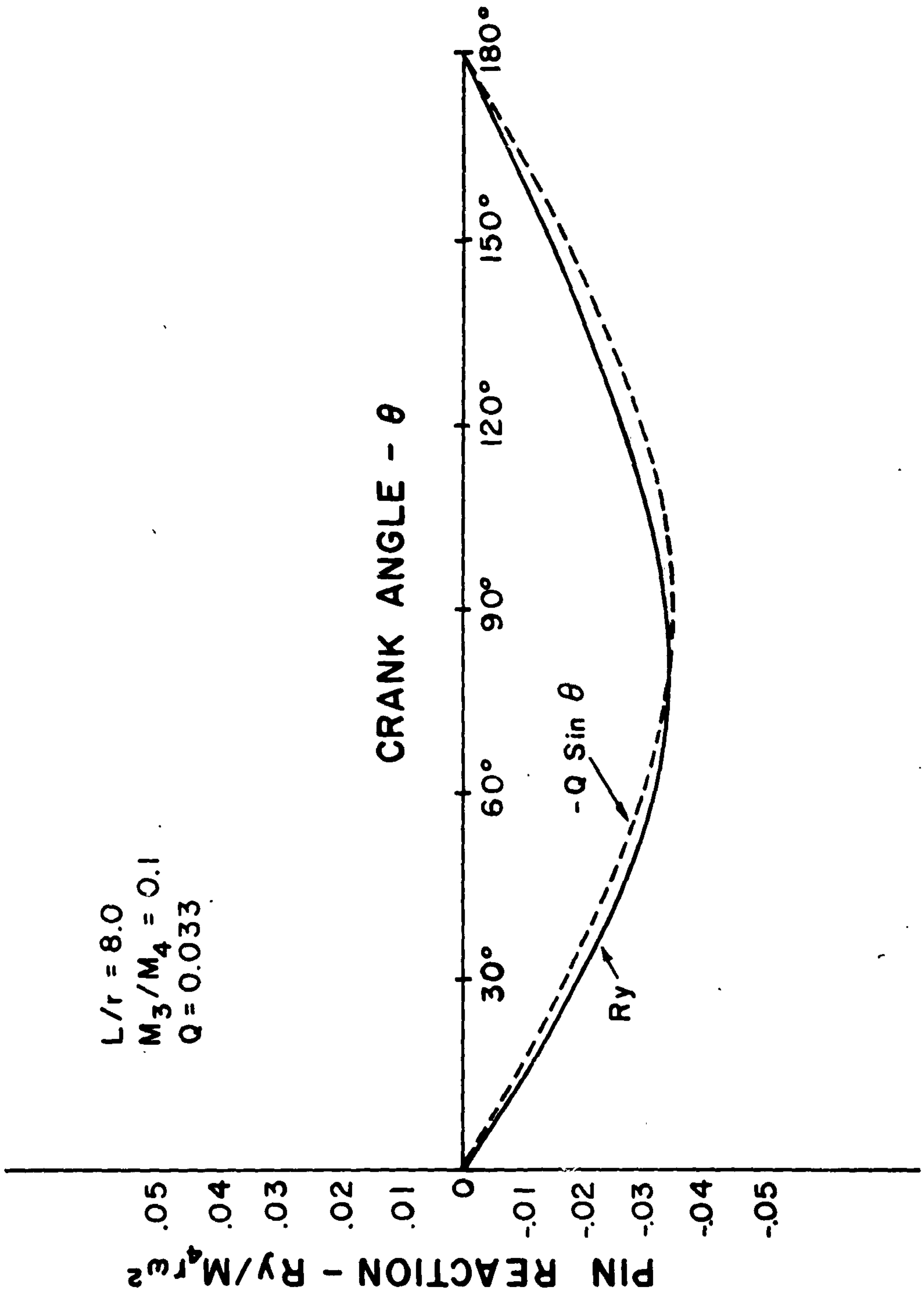


Figure 13. Pin Reaction Ry Compared with - Q sin θ

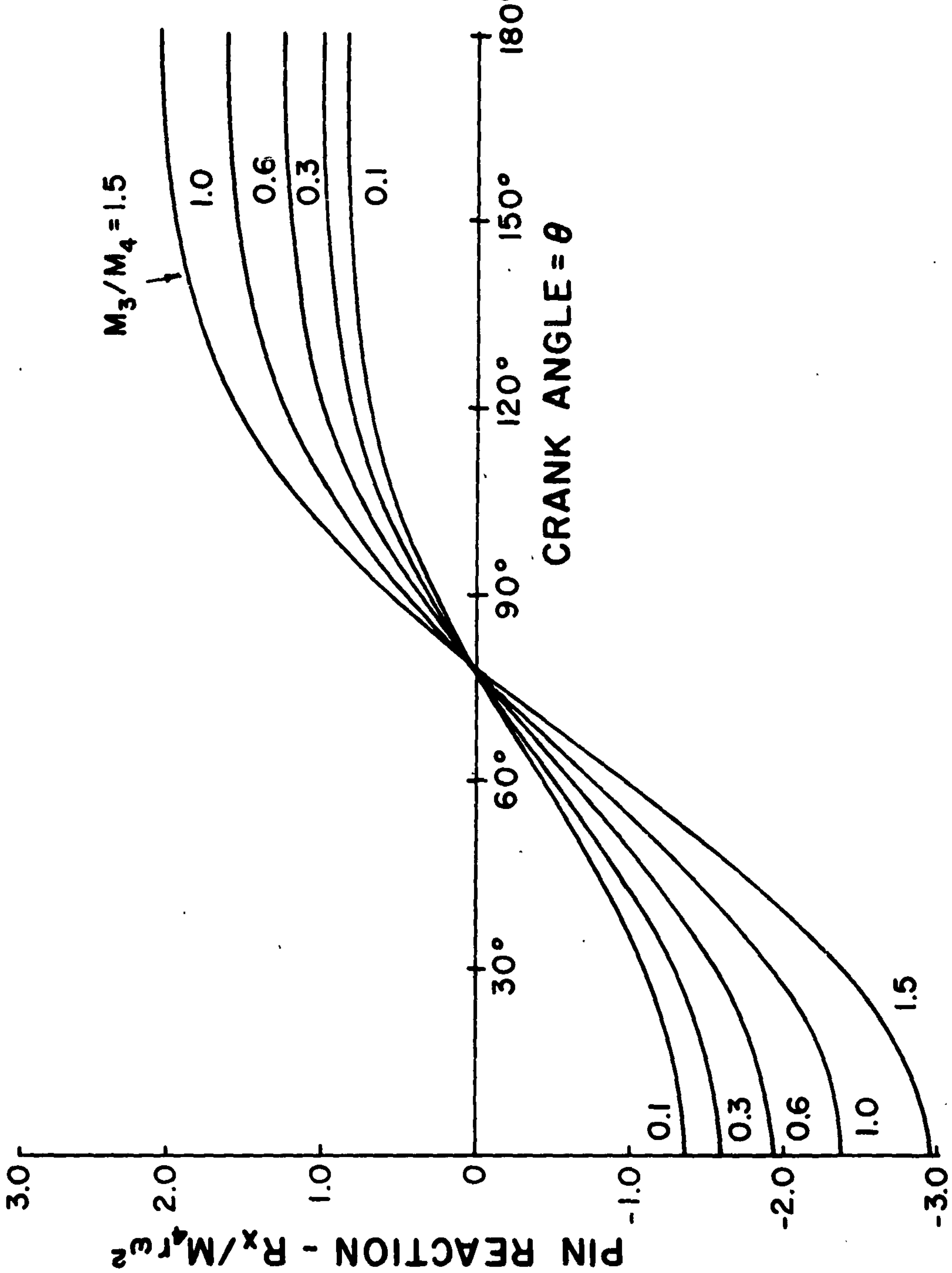


Figure 14. Pin Reaction R_x for $L/r = 4.0$

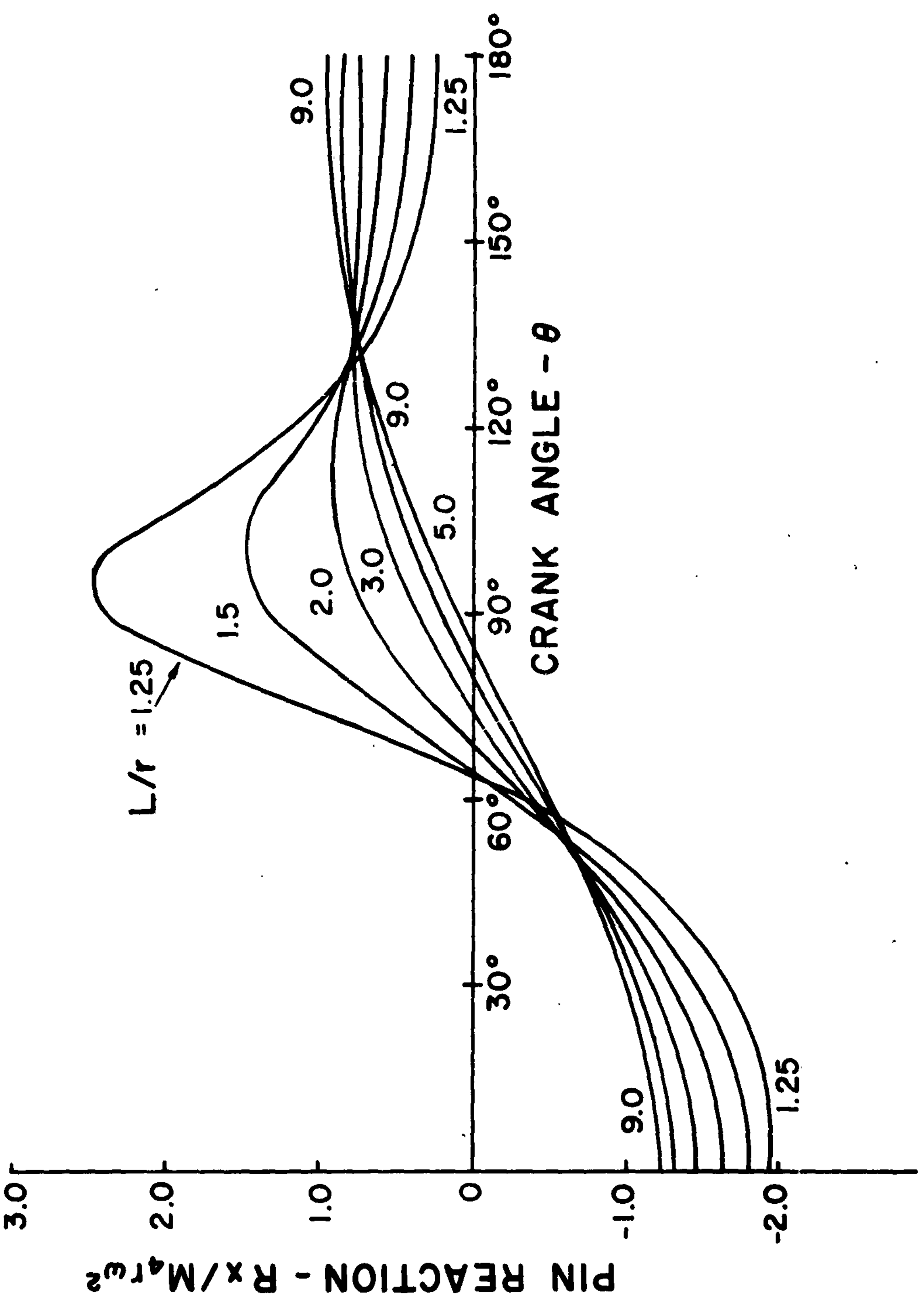


Figure 15. Pin Reaction R_x for $M_3/M_4 = 0.1$

approximations

$$R_x = -K\cos\theta, \qquad (17)$$

$$R_y = -Q\sin\theta,$$

seem to be valid when the r/L ratio is small ($r/L < \frac{1}{4}$) and the M_3/M_4 ratio is small ($M_3/M_4 < 1.5$). The value of the positive constant K can be approximated by

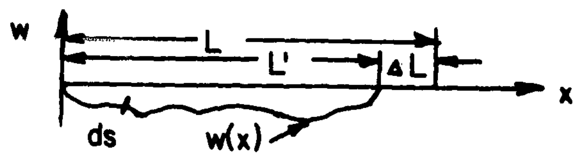
$$K = (M_3 + M_4)\, r\, \omega^2. \qquad (18)$$

The value of the positive constant Q was not required for the solution of the equation of motion but can be approximated as,

$$Q = \frac{M_3\, r\, \omega^2}{3}\ .$$

APPENDIX C

Derivation of Equations to Determine Variation in Piston Displacement.



If the assumption is made that the length of the beam L remains constant then

$$L = \int_0^{L'} d\,S = \int_0^{L'} \sqrt{1 + (w')^2}\, d\,x.$$

Substituting the binomial expansion for the radical and neglecting all but the first term yields

$$L = \int_0^{L'} [1+\tfrac{1}{2}(w')^2 + \tfrac{1}{8}(w')^4 + ---]dx \cong L' + \int_0^{L'} \tfrac{1}{2}(w')^2\, dx.$$

Since $\Delta L = L - L'$

and since ΔL will be small, it can be approximated by

$$\Delta L = \int_0^{L} \tfrac{1}{2}(w')^2\, dx.$$

Substituting the derivative of expression (47) into the above yields

$$\Delta L = \int_0^{L} \tfrac{1}{2}\left(\frac{q\pi}{L}\right)^2 \cos^2 \frac{\pi x}{L}\, dx = \frac{\pi^2 q^2}{4L} \qquad (1c)$$

where q is the amplitude $w(\frac{L}{2},\theta)$ taken from Figures 6, 7, 9, 10 or 11.

The position of the piston during a vibration can be written as

$$X_B' = r\cos\theta + L'(1 - (\tfrac{r}{L'})^2 \sin^2\theta)^{\frac{1}{2}}$$

Since the vibration has been assumed small $r/L \cong r/L'$. Therefore, substituting r/L into the above equation and subtracting it from equation (1) yields

$$\Delta X_B = X_B - X_B' = \Delta L(1 - (\tfrac{r}{L})^2 \sin^2\theta)^{\frac{1}{2}}$$

the variation in piston displacement.